Reflections on delta lenses

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15 June 2023

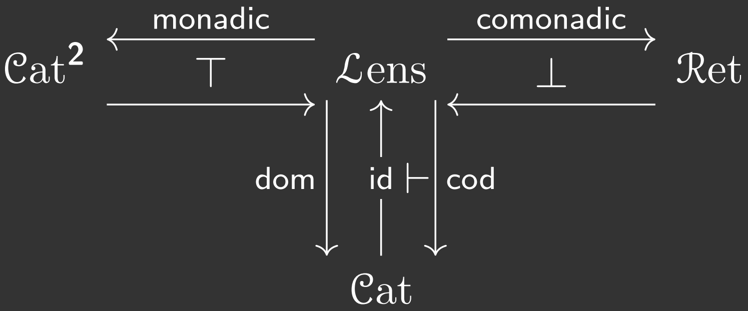
Figure: The Cadbury chocolate factory in Tasmania, Australia is the largest in the southern hemisphere and has been operating since 1921.

Motivation

- A *system* can be seen as a *category*, whose objects are the states of the system, and whose morphisms are the updates or transitions between system states.
- A bidirectional transformation between systems is a *consistency relation* between the states together with a *synchronisation specification* which propagates updates.
- Lenses are a algebraic model for bidirectional transformations: a lens *focuses* on some *view* of a *source* system.
- Lenses, and delta lenses in particular, have applications in fields such as model-driven engineering, database sychronisation, machine learning, optimal transport, functional programming, etc.

This talk: Delta lenses are *morphisms* between categories; what is the (double) category theory of lenses, and how can this be useful for applications?

The big picture



Outline of the talk

1. Introduction to (delta) lenses
2. The relationship with retrofunctors
3. Delta lenses as (co)algebras for a (co)monad
4. Composition and double categories

Part 1: Introduction to (delta) lenses

An abridged history of lenses[1]

- 2004: Benjamin Pierce (+ coauthors) introduces the notion of lens between sets in a talk at the *New England Programming Languages Symposium*. The paper *Combinators for bidirectional tree transformations* published in 2007.
- 2011: Diskin, Xiong, and Czarnecki introduce delta lenses between categories as a generalisation of lenses between sets, motivated by model-driven engineering.
- 2013: Johnson and Rosebrugh publish first serious study of delta lenses using category theory, show that they generalise split opfibrations and arise as certain algebras for a semi-monad. Research programme continues for the next 10 years.
- 2017: Ahman and Uustalu demonstrate link between delta lenses and cofunctors.

[1] See references at the end of slides.

Classical state-based lenses (between sets)

A state-based lens $A \nrightarrow B$ is pair of functions $f \colon A \to B$ and $p \colon A \times B \to A$,

$$
\begin{array}{ccccc}
A & & a & \longrightarrow & p(a, b) \\
{\scriptstyle (f,\, p)} \big\downarrow & & \vdots & & \vdots \\
B & & fa & \longrightarrow & b
\end{array}
$$

which satisfy the following three axioms:

1. $fp(a, b) = b$,
2. $p(a, fa) = a$,
3. $p(p(a, b), b') = p(a, b')$.

Proposition: State-based lenses are equivalent to product projections $\pi \colon X \times B \to B$.

Delta lenses

A delta lens $A \nrightarrow B$ is a functor $f\colon A \to B$ equipped with a lifting operation φ

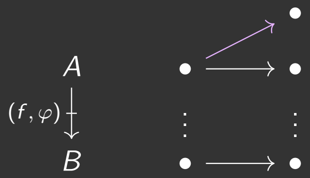
$$
\begin{array}{ccc}
A & \qquad & a \xrightarrow{\varphi(a,u)} p(a,u) \\
{\scriptstyle (f,\varphi)}\big\downarrow & & \vdots \qquad\qquad\quad \vdots \\
B & \qquad & fa \xrightarrow{\quad u \quad} b
\end{array}
$$

where $p(a,u) := \mathrm{cod}(\varphi(a,u))$, satisfying the following three axioms:

1. $f\varphi(a,u) = u$,
2. $\varphi(a, 1_{fa}) = 1_a$,
3. $\varphi(a, v \circ u) = \varphi(p(a,u), v) \circ \varphi(a,u)$.

Examples of delta lenses

- A state-based lens is a delta lens between *codiscrete categories*.
- A split opfibration is a delta lens such that the chosen lifts $\varphi(a, u)$ are *opcartesian*.
- A discrete opfibration is a delta lens such that $\varphi(a, fw) = w$, i.e. *unique* lift.
- A monoid homomorphism with a chosen right inverse is a delta lens between monoids. Also, any bijective-on-objects functor with a chosen right inverse.
- Consider the categories $A = \{\bullet \leftarrow \bullet \rightarrow \bullet\}$ and $B = \{\bullet \rightarrow \bullet\}$. There is a delta lens



where the purple arrow is the non-trivial chosen lift.

Part 2: The relationship with retrofunctors

Retrofunctors (better known as cofunctors)

A retrofunctor $A \nrightarrow B$ is a functor $f\colon A_0 \to B$ equipped with a lifting operation φ

$$
\begin{array}{ccccc}
A & \quad & a & \xrightarrow{\varphi(a,u)} & p(a,u) \\
{\scriptstyle (f,\varphi)} \big\downarrow\!\!\!\!+ & & \vdots & & \vdots \\
B & & fa & \xrightarrow{\quad u \quad} & b
\end{array}
$$

where $p(a,u) \coloneqq \mathrm{cod}(\varphi(a,u))$, satisfying the following three axioms:

1. $fp(a,u) = \mathrm{cod}(u)$,
2. $\varphi(a, 1_{fa}) = 1_a$,
3. $\varphi(a, v \circ u) = \varphi(p(a,u), v) \circ \varphi(a,u)$.

The *identity retrofunctor* on A consists of the functor $i_A\colon A_0 \to A$ and the lifting operation given by projection $\pi(a,u) = u$.

Retrofunctors as certain spans of functors

Every retrofunctor $(f, \varphi) \colon A \nrightarrow B$ induces a span of functors, as depicted below, such that $\overline{\varphi}$ is bijective-on-objects and $\overline{f}$ is a discrete opfibration.

$$A \xleftarrow{\quad \overline{\varphi} \quad} \Lambda(f, \varphi) \xrightarrow{\quad \overline{f} \quad} B$$

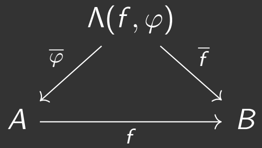
$$\begin{array}{ccccc} a & & a & & fa \\ \Big\downarrow {\scriptstyle \varphi(a, u)} & & \Big\downarrow {\scriptstyle (a, u)} & & \Big\downarrow {\scriptstyle u} \\ p(a, u) & & p(a, u) & & b \end{array}$$

Conversely, every such span of this kind induces a retrofunctor, and *isomorphic* spans yield the *same* retrofunctor. This determines an equivalence of (double) categories!

Delta lenses as compatible functors and retrofunctors

A delta lens consists of a functor $f \colon A \to B$ and a retrofunctor $(f', \varphi) \colon A \nrightarrow B$ such that $f' = f \circ i_A = i_B \circ f_0$ (i.e. f and f' agree on objects) and $f\varphi(a, u) = u$.

Corollary: Every delta lens $(f, \varphi) \colon A \nrightarrow B$ induces a commutative diagram in $\mathcal{C}\mathrm{at}$, as depicted below, such that $\overline{\varphi}$ is bijective-on-objects and $\overline{f}$ is a discrete opfibration.



Compatible squares of functors and retrofunctors

A compatible square of functors and retrofunctors is given by a diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle (f,\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle (g,\psi)} \\ B & \xrightarrow[k]{} & D \end{array}$$

such that $g \circ h_0 = k \circ f$ and $h\varphi(a, u) = \psi(ha, ku)$ for all $(a \in A, u \colon fa \to b \in B)$.

A delta lens is a compatible square of the form:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle (f',\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle (i_B,\pi)} \\ B & \xrightarrow[1_B]{} & B \end{array}$$

Forgetful functors

- Let $\mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at}) = \mathcal{C}\mathrm{at}^{\mathbf{2}}$ denote the category whose objects are functors and whose morphisms are commutative squares.
- Let $\mathcal{R}\mathrm{et}$ denote the category whose objects are retrofunctors and whose morphisms are compatible squares.
- Let $\mathcal{L}\mathrm{ens}$ denote the category whose objects are delta lenses and whose morphisms are compatible squares such that the underlying square of functors commutes.
- There are canonical forgetful functors:

$$\mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at}) \xleftarrow{\quad U \quad} \mathcal{L}\mathrm{ens} \xrightarrow{\quad V \quad} \mathcal{R}\mathrm{et}$$

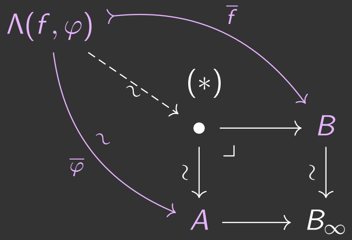
$$A \xrightarrow{f} B \qquad\qquad A \xrightarrow{(f, \varphi)} B \qquad\qquad A \xrightarrow{(f \circ i_A, \varphi)} B$$

Part 3: Delta lenses as (co)algebras for a (co)monad

Delta lenses as coalgebras for a comonad

The functor $V \colon \mathcal{L}\mathrm{ens} \to \mathcal{R}\mathrm{et}$ is *comonadic*.

We can construct the right adjoint $\mathcal{R}\mathrm{et} \to \mathcal{L}\mathrm{ens}$ *diagrammatically*. We use the notation $\xrightarrow{\sim}$ for bijective-on-objects functors and $\rightarrowtail$ for discrete opfibrations. Given a retrofunctor (considered as a span), the cofree delta lens is given by $(*)$.



The cofree delta lens on a retrofunctor

The cofree delta lens $\pi_2 \colon \mathrm{Cofree}(f, \varphi) \nrightarrow B$ on a retrofunctor $(f, \varphi) \colon A \nrightarrow B$ has domain whose:

- objects are those of A;
- morphisms $a \to a'$ are pairs $(w \colon a \to a' \in A, u \colon fa \to fa' \in B)$.

The underlying functor π_2 of the cofree delta lens sends $(w, u) \colon a \to a'$ to $u \colon fa \to fa'$.

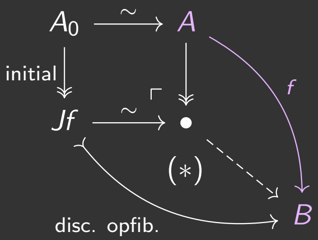
The lifting operation for the cofree delta lens is given by:

$$
\begin{array}{c}
\mathrm{Cofree}(f, \varphi) \\
\Big\downarrow\!\!\!\!+ \\
B
\end{array}
\qquad\qquad
\begin{array}{ccc}
a & \xrightarrow{(\varphi(a, u), u)} & p(a, u) \\
\vdots & & \vdots \\
fa & \xrightarrow{\quad u \quad} & b
\end{array}
$$

Delta lenses as algebras for a monad

The functor $U\colon \mathcal{L}\mathrm{ens} \to \mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at})$ is *monadic*.

The left adjoint $\mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at}) \to \mathcal{L}\mathrm{ens}$ may be constructed diagrammatically using the comprehensive factorisation system on $\mathcal{C}\mathrm{at}$. We use the notation $\twoheadrightarrow$ for *inital functors*. Given a functor, the free delta lens is given by $(*)$.



The free delta lens on a functor

The free delta lens $\mathrm{cod}\colon \mathrm{Free}(f) \nrightarrow B$ on a functor $f\colon A \to B$ has domain whose:

- objects are pairs $(a \in A, u\colon fa \to b \in B)$;
- morphisms $(a_1, u_1) \to (a_2, u_2)$ are given by the following two sorts:

$$
\begin{array}{ccc}
a_1 & = & a_2 \\
 & & \\
fa_1 & = & fa_2 \\
{\scriptstyle u_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle u_2} \\
b_1 & \xrightarrow{\;v\;} & b_2
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
a_1 & = & a_1 & \xrightarrow{\;w\;} & a_2 & = & a_2 \\
 & & & & & & \\
fa_1 & = & fa_1 & \xrightarrow{\;fw\;} & fa_2 & = & fa_2 \\
{\scriptstyle u_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle u_2} \\
b_1 & \xrightarrow{\;v\;} & fa_1 & \xrightarrow{\;fw\;} & fa_2 & \xrightarrow{\;u_2\;} & b_2
\end{array}
$$

The underlying functor cod of the free delta lens sends these morphisms to $v\colon b_1 \to b_2$ and $u_2 \circ fw \circ v\colon b_1 \to b_2$, respectively. The chosen lifts are morphisms of the first sort.

The algebraic weak factorisation system for delta lenses

Delta lenses are the R-algebras of an *algebraic weak factorisation system*.

Corollary: Consider a commutative diagram $(\dagger)$ in $\mathcal{C}\mathrm{at}$ such that g is a delta lens.

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\big\downarrow & (\dagger) & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array} \qquad\qquad \begin{array}{ccc} fqb_1 & \xrightarrow{fqu} & fqb_2 \\ {\scriptstyle \varepsilon_{b_1}}\big\downarrow\big\uparrow{\scriptstyle \overline{q}u} & (*) & \big\downarrow{\scriptstyle \varepsilon_{b_2}} \\ b_1 & \xrightarrow[u]{} & b_2 \end{array}$$

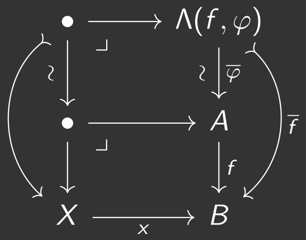
Suppose f has a left-adjoint-right-inverse $q \colon B \to A$ with counit ε such that if $q(u \colon b_1 \to b_2) \neq 1$, there is a specified $\overline{q}u \colon b_1 \to fqb_1$ such that $(*)$ "commutes".

Then there exists a canonical functor $j \colon B \to C$ such that $g \circ j = k$ and $j \circ f = h$.

Change of base: pullback back along a functor

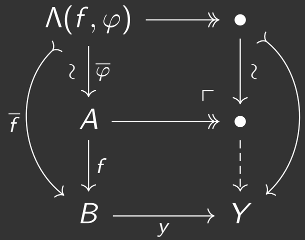
Let $\mathrm{cod}\colon \mathcal{L}\mathrm{ens} \to \mathcal{C}\mathrm{at}$ be the functor which assigns a delta lens to its codomain, and let $\mathcal{L}\mathrm{ens}(B)$ denote the fibre over B.

The functor $\mathrm{cod}\colon \mathcal{L}\mathrm{ens} \to \mathcal{C}\mathrm{at}$ is a fibration, yielding change of base functors $\Delta_x\colon \mathcal{L}\mathrm{ens}(B) \to \mathcal{L}\mathrm{ens}(X)$ for each functor $x\colon X \to B$.



Change of base: pushing forward along a functor

The functor $\mathrm{cod}\colon \mathcal{L}\mathrm{ens} \to \mathcal{C}\mathrm{at}$ is an opfibration, yielding change of base functors $\Sigma_y\colon \mathcal{L}\mathrm{ens}(B) \to \mathcal{L}\mathrm{ens}(Y)$ for each functor $y\colon B \to Y$.

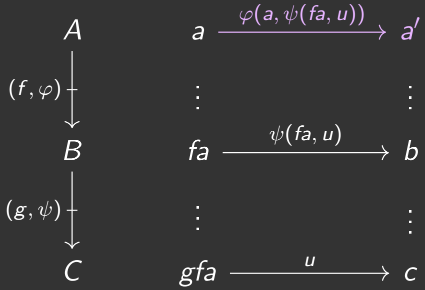


If $y\colon B \to Y$ is a discrete opfibration, then Σ_y is precisely *composition* of delta lenses.

Part 4: Composition and double categories

Composing retrofunctors

Given retrofunctors $(f, \varphi)\colon A \nrightarrow B$ and $(g, \psi)\colon B \nrightarrow C$, their composite $A \nrightarrow C$ consists of the functor $g \circ f_0\colon A_0 \to C$ and the lifting operation as depicted below.



Composition is strictly unital and associative, and extends to compatible squares.

Composing delta lenses

The composition of delta lenses is determined by the respective composition of their underlying functors and retrofunctors.

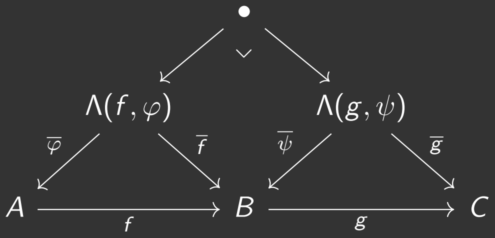
$$
\begin{array}{ccccc}
A & \xrightarrow{\;f\;} & B & \xrightarrow{\;g\;} & C \\
{\scriptstyle (f',\varphi)}\big\downarrow & & {\scriptstyle (i_B,\pi)}\big\downarrow & & \big\downarrow{\scriptstyle (i_C,\pi)} \\
B & \xrightarrow{\;1_B\;} & B & \xrightarrow{\;g\;} & C \\
{\scriptstyle (g',\psi)}\big\downarrow & & {\scriptstyle (g',\psi)}\big\downarrow & & \big\downarrow{\scriptstyle (i_C,\pi)} \\
C & \xrightarrow[\;1_C\;]{} & C & \xrightarrow[\;1_C\;]{} & C
\end{array}
$$

The *identity delta lens* consists of the identity functor and identity retrofunctor.

Composition is strictly unital and associative, and extends to compatible squares.

Diagrammatic composition via pullbacks

Given delta lenses $(f, \varphi)\colon A \nrightarrow B$ and $(g, \psi)\colon B \nrightarrow C$, their composite may be computed diagrammatically using pullbacks.



Delta lenses also admit a nice diagrammatic composition as R-algebras for an AWFS via canonical lifts against cofree L-coalgebras.

Double categories

A (strict) double category is an internal category in $\mathcal{CAT}$.

$$\mathbb{D} \quad : \quad \mathcal{D}_0 \underset{\text{cod}}{\overset{\text{dom}}{\leftarrow \!\! \xrightarrow{\text{id}} \!\! \leftarrow}} \mathcal{D}_1 \xleftarrow{\odot} \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 = \mathcal{D}_2$$

- $\mathcal{D}_0$ - the category of objects and horizontal morphisms;
- $\mathcal{D}_1$ - the category of vertical morphisms and cells;

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ f \downarrow & \alpha & \downarrow g \\ B & \xrightarrow[k]{} & D \end{array}$$

A double category of functors and delta lenses

Let $\mathbb{L}\mathrm{ens}$ denote the double category whose:

- objects are categories;
- horizontal morphisms are functors;
- vertical morphisms are delta lenses;
- cells are suitable compatible squares.

The corresponding internal category in $\mathcal{CAT}$ is denoted as follows.

$$\mathbb{L}\mathrm{ens} \quad : \quad \mathcal{C}\mathrm{at} \underset{\text{cod}}{\overset{\text{dom}}{\leftleftarrows}} \xrightarrow{\text{id}} \mathcal{L}\mathrm{ens} \xleftarrow{\odot} \mathcal{L}\mathrm{ens} \times_{\mathcal{C}\mathrm{at}} \mathcal{L}\mathrm{ens}$$

This double category provides a "nice place" to study delta lenses.

Some properties of the double category $\mathbb{L}\text{ens}$

- The identity-assigning functor $\text{id}\colon \mathcal{C}\text{at} \to \mathcal{L}\text{ens}$ has both a left adjoint $\text{cod}\colon \mathcal{L}\text{ens} \to \mathcal{C}\text{at}$ and a right adjoint $\Lambda\colon \mathcal{L}\text{ens} \to \mathcal{C}\text{at}$ (the tabulator).
- The functor $\text{cod}\colon \mathcal{L}\text{ens} \to \mathcal{C}\text{at}$ is a bifibration. (Is it a trifibration?)
- A monoidal double category in two different ways: product and coproduct.
- Companions are discrete opfibrations; conjoints are isomorphisms.
- A thin double category: each cell is determined by its boundary.
- The right-connected completion of the double category $\mathbb{R}\text{et}$ of categories, functors, retrofunctors, and compatible squares. (Links with AWFS)
- Cofibrantly generated by a small double category. (See bonus slides)
- Admits canonical double functors to $U\colon \mathbb{L}\text{ens} \to \mathbb{S}\text{q}(\mathcal{C}\text{at})$ and $V\colon \mathbb{L}\text{ens} \to \mathbb{R}\text{et}$.

Summary and further work

Delta lenses are a rich structure with useful and interesting characterisations: compatible functors/retrofunctors, (co)algebras for a (co)monad, AWFS, etc.

- Grothendieck construction for delta lenses using (split) multi-valued functions.
- Delta lenses in internal and enriched category theory (e.g. weighted lenses).
- Theory of symmetric lenses as spans of delta lenses (applications to *bisimulations*).
- Generalisation as morphisms of monads in double categories using *retrocells*.
- Lenses as compatible vertical/horizontal morphisms in a double category — the right-connected completion of a double category.

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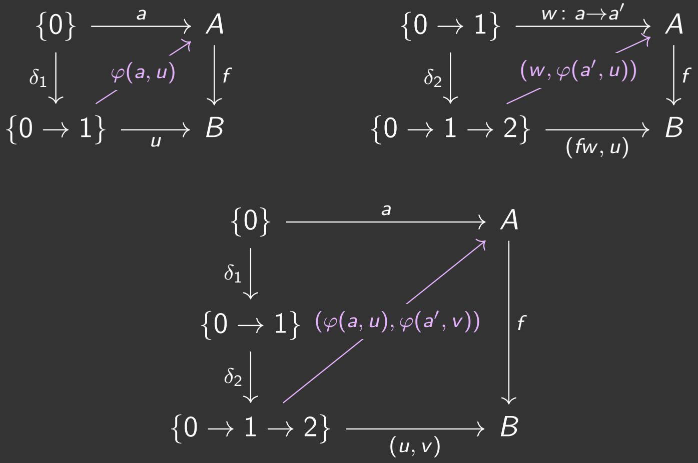
Bonus slides: Cofibrant generation by a small double category

Let $\mathbb{J}$ denote the small double category whose:

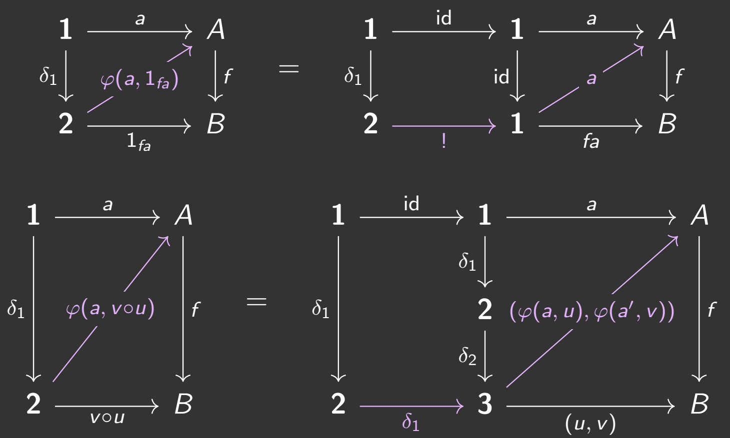
- objects are the ordinals $\mathbf{1} = \{0\}$, $\mathbf{2} = \{0 < 1\}$ and $\mathbf{3} = \{0 < 1 < 2\}$;
- horizontal morphism are all order preserving maps;
- vertical morphisms are freely generated by $\delta_1 \colon \mathbf{1} \to \mathbf{2}$ and $\delta_2 \colon \mathbf{2} \to \mathbf{3}$;
- cells are freely generated by the following commutative squares.

$$
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{\delta_0} & \mathbf{2} \\
{\scriptstyle \delta_1}\downarrow & & \downarrow{\scriptstyle \delta_2} \\
\mathbf{2} & \xrightarrow[\delta_0]{} & \mathbf{3}
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{\mathrm{id}} & \mathbf{1} \\
{\scriptstyle \delta_1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathbf{2} & \xrightarrow[!]{} & \mathbf{1}
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{\mathrm{id}} & \mathbf{1} \\
 & & \downarrow{\scriptstyle \delta_1} \\
{\scriptstyle \delta_1}\downarrow & & \mathbf{2} \\
 & & \downarrow{\scriptstyle \delta_2} \\
\mathbf{2} & \xrightarrow[\delta_1]{} & \mathbf{3}
\end{array}
$$

A delta lens is a functor $f\colon A \to B$ equipped with lifting operations . . .



. . . which satisfy the following equations.



The double category $\mathbb{L}\mathrm{ens}$ is cofibrantly generated by the double functor $\mathbb{J} \to \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at})$.