The right-connected completion of a double category

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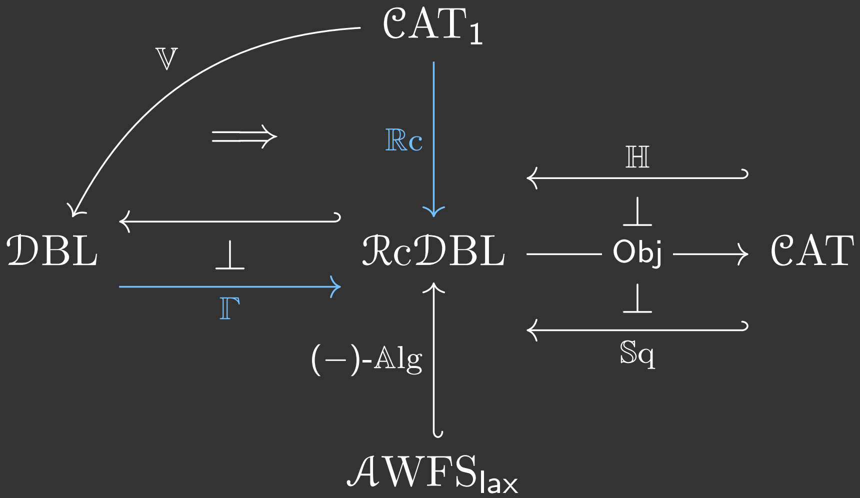
Motivation

- Algebraic weak factorisation systems (AWFS) are a generalisation of orthogonal factorisation systems (OFS).
- John Bourke and Richard Garner determined a characterisation of AWFS in terms of double categories with certain properties.
- This talk focuses on one of these properties, right-connectness, where each vertical morphism has an *underlying* horizontal morphism:

$$
\begin{array}{c}
A \\
f \big\downarrow \\
B
\end{array}
\quad \longmapsto \quad
\begin{array}{ccc}
A & \xrightarrow{Uf} & B \\
f \big\downarrow & \rho_f & \big\downarrow \mathrm{id}_B \\
B & \xrightarrow[1_B]{} & B
\end{array}
$$

- Our goal is to construct the right-connected completion of a double category, with a view towards generating new examples and a better understanding of AWFS.

The big picture



Outline of the talk

1. Introducing (right-connected) double categories

2. Algebraic weak factorisation systems

3. The right-connected completion

Part 1: Introducing (right-connected) double categories

Double categories

A double category is a (unital) pseudo category object in the 2-category $\mathcal{CAT}$.

$$\mathbb{D} \quad : \quad \mathcal{D}_0 \; \begin{array}{c} \xleftarrow{\text{dom}} \\ \xrightarrow{\text{id}} \\ \xleftarrow[\text{cod}]{} \end{array} \; \mathcal{D}_1 \xleftarrow{\quad \odot \quad} \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 = \mathcal{D}_2$$

- $\mathcal{D}_0$ - the category of objects and horizontal morphisms;
- $\mathcal{D}_1$ - the category of vertical morphisms and cells;

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ f \downarrow\!\!\!\!+ & \alpha & +\!\!\!\!\downarrow g \\ B & \xrightarrow[k]{} & D \end{array}$$

Right-connected double categories

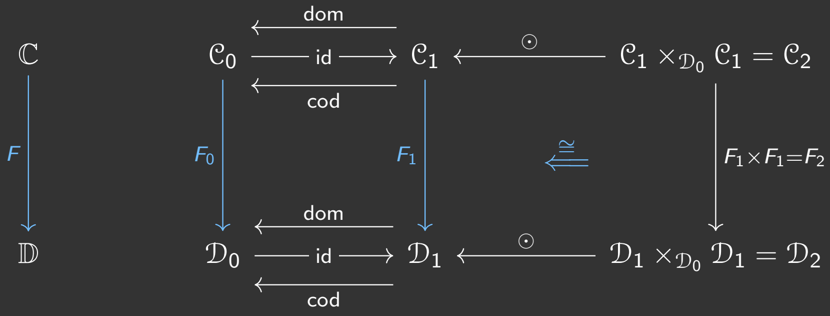
A double category is right-connected if its identity-assigning map is *right adjoint* to its codomain-assigning map.

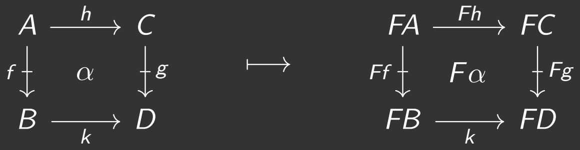
$$\mathcal{D}_0 \underset{\text{cod}}{\overset{\text{id}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathcal{D}_1$$

For every vertical morphism $f : A \nrightarrow B$, there is a *underlying* horizontal morphism $Uf : A \to B$ together with a universal cell ρ_f:

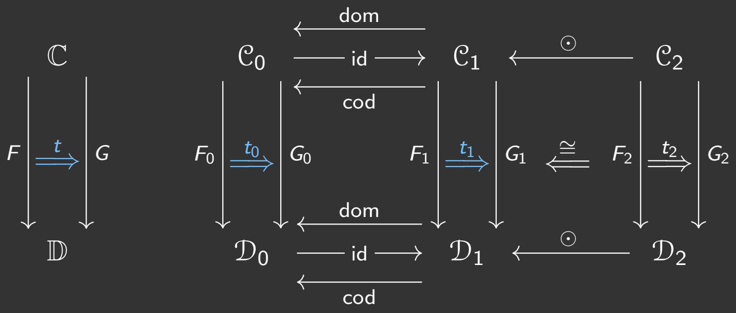
$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & \forall\,\alpha & \downarrow{\scriptstyle \text{id}_C} \\ B & \xrightarrow[k]{} & C \end{array} \quad \overset{\exists!}{=} \quad \begin{array}{ccccc} A & \xrightarrow{Uf} & B & \xrightarrow{k} & C \\ {\scriptstyle f}\downarrow & \rho_f & \downarrow{\scriptstyle \text{id}_B} & \text{id}_k & \downarrow{\scriptstyle \text{id}_C} \\ B & \xrightarrow[1_B]{} & B & \xrightarrow[k]{} & C \end{array}$$

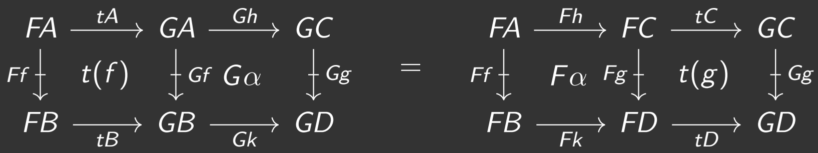
Unitary double functors





Horizontal transformations





The 2-category of (right-connected) double categories

Let $\mathcal{DBL}$ denote the 2-category whose:

- objects are double categories;
- morphisms are unitary double functors;
- 2-cells are horizontal transformations.

Let $\mathcal{R}c\mathcal{DBL}$ denote the restriction of $\mathcal{DBL}$ to the right-connected double categories.

Our goal of the talk is to construct a right 2-adjoint to the 2-functor:

$$\mathcal{R}c\mathcal{DBL} \hookrightarrow \mathcal{DBL}$$

... but first, some examples!

Example: The double category of squares

Let $\mathbb{S}\mathrm{q}(\mathcal{C})$ denote the *double category of (commutative) squares* in a category $\mathcal{C}$.

- objects are given by the objects in $\mathcal{C}$;
- horizontal and vertical morphisms are given by the morphisms in $\mathcal{C}$;
- cells are given by the commutative squares in $\mathcal{C}$.

Right-connectedness of $\mathbb{S}\mathrm{q}(\mathcal{C})$ is given by the unique factorisation:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1_C} \\ B & \xrightarrow[k]{} & C \end{array} \quad \overset{\exists!}{=\!=} \quad \begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{k} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1_B} & & \downarrow{\scriptstyle 1_C} \\ B & \xrightarrow[1_B]{} & B & \xrightarrow[k]{} & C \end{array}$$

A canonical forgetful double functor

Given a right-connected double category $\mathbb{D} = \langle \mathcal{D}_0, \mathcal{D}_1 \rangle$ there is a double functor:

$$\mathbb{D} \xrightarrow{\quad U \quad} \mathbb{S}\mathrm{q}(\mathcal{D}_0)$$

$$\begin{array}{ccc}
\mathcal{D}_1 & \xrightarrow{\quad U_1 \quad} & \mathcal{S}\mathrm{q}(\mathcal{D}_0) \\
\mathrm{dom}\downarrow \;\; \uparrow \mathrm{id} \vdash \;\; \downarrow \mathrm{cod} & & \mathrm{dom}\downarrow \;\; \uparrow \mathrm{id} \vdash \;\; \downarrow \mathrm{cod} \\
\mathcal{D}_0 & = \!=\!=\!=\!=\!= & \mathcal{D}_0
\end{array}$$

$$\begin{array}{ccc}
A & \xrightarrow{h} & C \\
f \downarrow\!\!\!\!+ & \alpha & \downarrow\!\!\!\!+ \, g \\
B & \xrightarrow[k]{} & D
\end{array}
\qquad \longmapsto \qquad
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
Uf \downarrow & & \downarrow Ug \\
B & \xrightarrow[k]{} & D
\end{array}$$

Example: The horizontal double category

Let $\mathbb{H}(\mathcal{C})$ denote the *horizontal double category* of a category $\mathcal{C}$, obtained by restricting the vertical morphisms in $\mathbb{S}\mathrm{q}(\mathcal{C})$ to the identities — it is also right-connected.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle 1_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

Generalising, let $\mathcal{C}$ be a category equipped with a wide subcategory $\mathcal{W}$. Then there is a right-connected double category $\mathbb{S}\mathrm{q}(\mathcal{C}, \mathcal{W})$ whose:

- objects and horizontal morphisms are given by $\mathcal{C}$;
- vertical morphisms are given by $\mathcal{W}$;
- cells are given by the commutative squares in $\mathcal{C}$.

An adjoint triple

We may define the following 2-functors:

- $\mathsf{Obj}\colon \mathcal{R}c\mathcal{D}\mathrm{BL} \to \mathcal{C}\mathrm{AT}, \quad \mathbb{D} = \langle \mathcal{D}_0, \mathcal{D}_1 \rangle \mapsto \mathcal{D}_0.$
- $\mathbb{S}\mathrm{q}\colon \mathcal{C}\mathrm{AT} \to \mathcal{R}c\mathcal{D}\mathrm{BL}, \quad \mathcal{C} \mapsto \mathbb{S}\mathrm{q}(\mathcal{C}).$
- $\mathbb{H}\colon \mathcal{C}\mathrm{AT} \to \mathcal{R}c\mathcal{D}\mathrm{BL}, \quad \mathcal{C} \mapsto \mathbb{H}(\mathcal{C}).$

There is an adjoint triple of 2-functors:

$$\begin{array}{ccc} & \xleftarrow{\quad \mathbb{H} \quad} & \\ & \perp & \\ \mathcal{R}c\mathcal{D}\mathrm{BL} & \xrightarrow{\quad \mathsf{Obj} \quad} & \mathcal{C}\mathrm{AT} \\ & \perp & \\ & \xleftarrow[\quad \mathbb{S}\mathrm{q} \quad]{} & \end{array}$$

The double functor $U\colon \mathbb{D} \to \mathbb{S}\mathrm{q}(\mathcal{D}_0)$ is the component at $\mathbb{D}$ of the unit of $\mathsf{Obj} \dashv \mathbb{S}\mathrm{q}$.

Example: The double category of split epimorphisms

Let $\mathbb{S}\mathrm{Epi}(\mathcal{C})$ denote the *double category of split epimorphisms* in $\mathcal{C}$.

- objects and horizontal morphisms are given by $\mathcal{C}$;
- vertical morphisms are split epimorphisms in $\mathcal{C}$;
- cells are given by diagrams in $\mathcal{C}$

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle\varphi}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle\psi}\uparrow\downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$$

such that $k \circ f = g \circ h$ and $h \circ \varphi = \psi \circ k$.

Right-connectedness:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle\varphi}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle 1_C}\uparrow\downarrow{\scriptstyle 1_C} \\ B & \xrightarrow[k]{} & C \end{array} \quad = \quad \begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{k} & C \\ {\scriptstyle\varphi}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle 1_B}\uparrow\downarrow{\scriptstyle 1_B} & & {\scriptstyle 1_C}\uparrow\downarrow{\scriptstyle 1_C} \\ B & \xrightarrow[1_B]{} & B & \xrightarrow[k]{} & C \end{array}$$

Digression: Delta lenses and split opfibrations

A delta lens is a functor $f \colon A \to B$ equipped with a lifting operation φ

$$
\begin{array}{c}
A \\
{\scriptstyle f}\big\downarrow \\
B
\end{array}
\qquad\qquad
\begin{array}{ccc}
a & \xrightarrow{\varphi(a,u)} & p(a,u) \\
\vdots & & \vdots \\
fa & \xrightarrow{\quad u \quad} & b
\end{array}
$$

where $p(a,u) := \mathrm{cod}(\varphi(a,u))$, satisfying the following three axioms:

1. $f\varphi(a,u) = u$,
2. $\varphi(a, 1_{fa}) = 1_a$,
3. $\varphi(a, v \circ u) = \varphi(p(a,u), v) \circ \varphi(a,u)$.

A split opfibration is a delta lens such that the chosen lifts $\varphi(a,u)$ are *opcartesian*.

Example: The double category of delta lenses / split opfibrations

Let $\mathbb{L}\mathrm{ens}$ denote the right-connected *double category of delta lenses* whose:

- objects are categories;
- horizontal morphisms are functors;
- vertical morphisms are delta lenses;
- cells are "compatible" squares

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle (f,\varphi)}\downarrow & & \downarrow{\scriptstyle (g,\gamma)} \\ B & \xrightarrow[k]{} & D \end{array}$$

such that $g \circ h = k \circ f$ and $h\varphi(a, u) = \gamma(ha, ku)$ for all $(a \in A, u \colon fa \to b \in B)$.

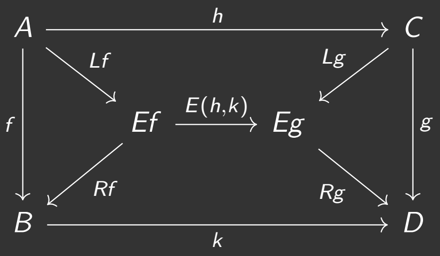
Let $\mathbb{S}\mathrm{Opf}$ be the restriction of $\mathbb{L}\mathrm{ens}$ to vertical morphisms which are split opfibrations.

Part 2: Algebraic weak factorisation systems

Algebraic weak factorisation systems (AWFS)

An algebraic weak factorisation system (L, R) on a category $\mathcal{C}$ consists of:

- A functorial factorisation on $\mathcal{C}$;



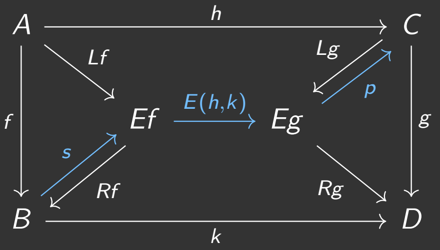
- A comonad (L, ϵ, Δ) and a monad (R, η, μ) on $\mathcal{C}^{\mathbf{2}} = \mathcal{S}\mathrm{q}(\mathcal{C})$;
- A distributive law $\delta \colon LR \Rightarrow RL$ of the comonad L over the monad R.

Lifting of coalgebras against algebras

Consider the L-coalgebra $\mathbf{f} = (f, s)$ and the R-algebra $\mathbf{g} = (g, p)$.

$$
\begin{array}{ccccc}
A & = & A & = & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Lf} & & \downarrow{\scriptstyle f} \\
B & \xrightarrow[s]{} & Ef & \xrightarrow[Rf]{} & B
\end{array}
=
\begin{array}{ccc}
A & = & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
B & \xrightarrow[1_B]{} & B
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{1_A} & C \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\
D & = & D
\end{array}
=
\begin{array}{ccccc}
C & \xrightarrow{Lg} & Eg & \xrightarrow{p} & C \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Rg} & & \downarrow{\scriptstyle g} \\
D & = & D & = & D
\end{array}
$$

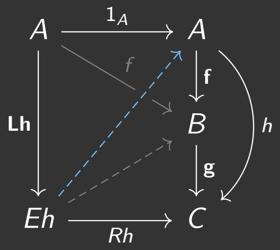
We may construct a canonical diagonal filler $\varphi_{\mathbf{f},\mathbf{g}}(h, k)$ for each commutative square:



The double category of R-algebras

Given an AWFS (L, R) on $\mathcal{C}$, there is a right-connected double category R-$\mathbb{A}\mathrm{lg}$ whose:

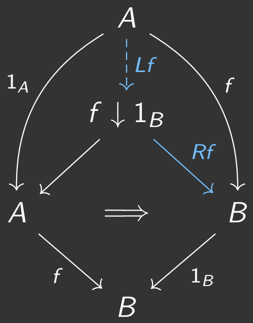
- objects and horizontal morphisms come from the category $\mathcal{C}$;
- vertical morphisms and cells come from the category of algebras R-$\mathcal{A}\mathrm{lg}$.



The R-algebra structure on h is given by $\varphi_{\mathbf{Lh},\mathbf{f}}\big(1_A, \varphi_{\mathbf{Lh},\mathbf{g}}(f, Rh)\big)\colon Eh \to A$.

Right-connected double categories arising from AWFS

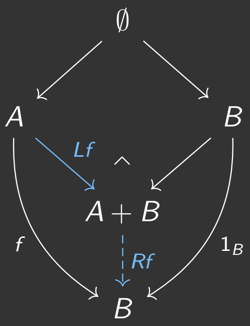
1. The double cat. $\mathbb{S}\mathrm{q}(\mathcal{C}, \mathcal{R})$, for an orthogonal factorisation system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$;



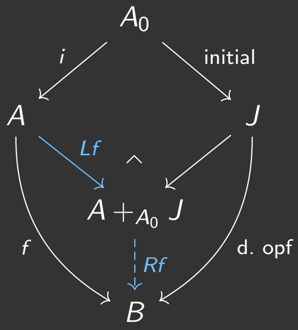
2. The double cat. $\mathbb{S}\mathrm{Opf} \cong R\text{-}\mathbb{A}\mathrm{lg}$ with AWFS constructed using comma categories.

More right-connected double categories arising from AWFS

3. If $\mathcal{C}$ has coproducts, then $\mathbb{SE}\mathrm{pi}(\mathcal{C}) \cong R\text{-}\mathbb{A}\mathrm{lg}$ with AWFS given by:



4. $\mathbb{L}\mathrm{ens} \cong R\text{-}\mathbb{A}\mathrm{lg}$ with AWFS given by:



$$J = \sum_{a \in A_0} fa/B$$

The Bourke-Garner characterisation theorem

Let $\mathcal{AWFS}_{\mathsf{lax}}$ denote the 2-category of AWFS and *lax morphisms*.

There is a diagram of 2-fully-faithful 2-functors:

$$\begin{array}{ccc} \mathcal{AWFS}_{\mathsf{lax}} & \xhookrightarrow{(-)\text{-}\mathbb{A}\mathrm{lg}} & \mathcal{DBL} \\ & {\scriptstyle (-)\text{-}\mathbb{A}\mathrm{lg}}\searrow \quad \nearrow & \\ & \mathcal{R}c\mathcal{DBL} & \end{array}$$

Bourke and Garner (2016): the 2-functor $(-)\text{-}\mathbb{A}\mathrm{lg}$ has in its *essential image* exactly those right-connected double categories $\mathbb{D} = \langle \mathcal{D}_0, \mathcal{D}_1 \rangle$ for which the canonical forgetful functor $U_1 \colon \mathcal{D}_1 \to \mathcal{S}\mathrm{q}(\mathcal{D}_0)$ is strictly monadic.

Part 3: The right-connected completion

The vertical double category

Let $\mathbb{V}(\mathcal{C})$ denote the *vertical double category* of a category $\mathcal{C}$, obtained by restricting the horizontal morphisms in $\mathbb{S}\mathrm{q}(\mathcal{C})$ to the identities — this is *not* right-connected, but does yield a 2-functor $\mathbb{V} \colon \mathcal{C}\mathrm{AT}_1 \to \mathcal{D}\mathrm{BL}$ from the *locally discrete* 2-category $\mathcal{C}\mathrm{AT}_1$.

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[1_B]{} & B \end{array}$$

<u>Question</u>: Does there exist an absolute left Kan lift of $\mathbb{V}$ along $\mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL} \hookrightarrow \mathcal{D}\mathrm{BL}$?

$$\begin{array}{ccc} & \mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL} & \\ & & \searrow \\ \mathcal{C}\mathrm{AT}_1 & \xrightarrow[\mathbb{V}]{} & \mathcal{D}\mathrm{BL} \end{array}$$

The "free" right-connected double category

Let $\mathbb{R}\mathrm{c}(\mathcal{C})$ denote *free right-connected double category* on $\mathbb{V}(\mathcal{C})$ for a category $\mathcal{C}$, obtained by restricting the (non-identity) cells in $\mathbb{S}\mathrm{q}(\mathcal{C})$ to those of the form:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1_B} \\ B & \xrightarrow[1_B]{} & B \end{array}$$

There is a 2-natural transformation

$$\begin{array}{ccc} & \mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL} & \\ {\scriptstyle \mathbb{R}\mathrm{c}}\nearrow & \Uparrow & \searrow \\ \mathcal{C}\mathrm{AT}_1 & \xrightarrow[\mathbb{V}]{} & \mathcal{D}\mathrm{BL} \end{array}$$

which characterises $\mathbb{R}\mathrm{c}$ as the $\mathbb{V}$-relative left 2-adjoint of $\mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL} \hookrightarrow \mathcal{D}\mathrm{BL}$.

Chasing isomorphisms of hom-categories

Recall that our goal is to construct a right 2-adjoint $\mathbb{\Gamma}$, called the right-connected completion of a double category:

$$\mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL} \underset{\mathbb{\Gamma}}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathcal{D}\mathrm{BL}$$

If $\mathbb{\Gamma}$ exists, then we have a natural isomorphism of hom-categories:

$$\mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL}\big(\mathbb{C}, \mathbb{\Gamma}(\mathbb{D})\big) \cong \mathcal{D}\mathrm{BL}\big(\mathbb{C}, \mathbb{D}\big)$$

From the relative 2-adjunction on the previous slide there is a natural isomorphism:

$$\mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL}\big(\mathbb{R}\mathrm{c}(\mathcal{C}), \mathbb{D}\big) \cong \mathcal{D}\mathrm{BL}\big(\mathbb{V}(\mathcal{C}), \mathbb{D}\big)$$

Combining these isomorphisms, we have the following:

$$\mathcal{D}\mathrm{BL}\big(\mathbb{V}(\mathcal{C}), \mathbb{\Gamma}(\mathbb{D})\big) \cong \mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL}\big(\mathbb{R}\mathrm{c}(\mathcal{C}), \mathbb{\Gamma}(\mathbb{D})\big) \cong \mathcal{D}\mathrm{BL}\big(\mathbb{R}\mathrm{c}(\mathcal{C}), \mathbb{D}\big)$$

The right-connected completion, abstractly

For any double category $\mathbb{D} = \langle \mathcal{D}_0, \mathcal{D}_1 \rangle$, there exists canonical isomorphisms

$$\mathcal{D}_0 \cong \mathcal{DBL}\big(\mathbb{V}(\mathbf{1}), \mathbb{D}\big) \qquad \mathcal{D}_1 \cong \mathcal{DBL}\big(\mathbb{V}(\mathbf{2}), \mathbb{D}\big)$$

where $\mathbf{1}$ is the *terminal category*, and $\mathbf{2} = \{0 \to 1\}$ is the *interval category*.

> The right-connected completion $\Gamma(\mathbb{D}) = \langle \Gamma_0(\mathbb{D}), \Gamma_1(\mathbb{D}) \rangle$ of a double category $\mathbb{D} = \langle \mathcal{D}_0, \mathcal{D}_1 \rangle$ is determined by the following isomorphisms:
>
> $$\Gamma_0(\mathbb{D}) \cong \mathcal{DBL}\big(\mathbb{R}\mathrm{c}(\mathbf{1}), \mathbb{D}\big) \cong \mathcal{D}_0 \qquad \Gamma_1(\mathbb{D}) \cong \mathcal{DBL}\big(\mathbb{R}\mathrm{c}(\mathbf{2}), \mathbb{D}\big)$$

Therefore, the objects and horizontal morphisms of $\Gamma(\mathbb{D})$ are given by those of $\mathbb{D}$.

What are the vertical morphisms and cells?

The right-connected completion, concretely

A vertical morphism in the right-connected completion $\Gamma(\mathbb{D})$ is given by a unitary double functor $\mathbb{R}\mathrm{c}(\mathbf{2}) \to \mathbb{D}$ which is determined by a triple:

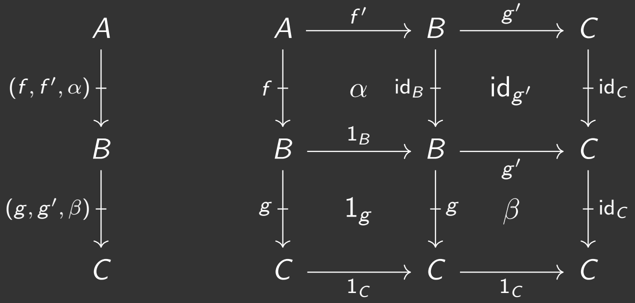
$$
\begin{array}{c}
A \\
{\scriptstyle (f,f',\alpha)}\big\downarrow \\
B
\end{array}
\qquad \rightsquigarrow \qquad
\begin{array}{ccc}
A & \xrightarrow{f'} & B \\
{\scriptstyle f}\big\downarrow & \alpha & \big\downarrow{\scriptstyle \mathrm{id}_B} \\
B & \xrightarrow[1_B]{} & B
\end{array}
$$

A cell in $\Gamma(\mathbb{D})$ is given by a horizontal transformation $(f, f', \alpha) \Rightarrow (g, g', \beta)$ of unitary double functors $\mathbb{R}\mathrm{c}(\mathbf{2}) \to \mathbb{D}$ which is determined by a cell θ in $\mathbb{D}$ such that:

$$
\begin{array}{ccccc}
A & \xrightarrow{h} & C & \xrightarrow{g'} & D \\
{\scriptstyle f}\big\downarrow & \theta & \big\downarrow{\scriptstyle g} & \beta & \big\downarrow{\scriptstyle \mathrm{id}_D} \\
B & \xrightarrow[k]{} & D & \xrightarrow[1_D]{} & D
\end{array}
\quad = \quad
\begin{array}{ccccc}
A & \xrightarrow{f'} & B & \xrightarrow{k} & D \\
{\scriptstyle f}\big\downarrow & \alpha & {\scriptstyle \mathrm{id}_B}\big\downarrow & \mathrm{id}_k & \big\downarrow{\scriptstyle \mathrm{id}_D} \\
B & \xrightarrow[1_B]{} & B & \xrightarrow[k]{} & D
\end{array}
$$

Vertical composition in $\Gamma(\mathbb{D})$

A *composable pair of vertical morphisms* in $\Gamma(\mathbb{D})$ is given by a *strict* double functor $\mathbb{R}\mathrm{c}(\mathbf{3}) \to \mathbb{D}$, and vertical composition is determined by pre-composing with the double functor $\mathbb{R}\mathrm{c}(d_1)\colon \mathbb{R}\mathrm{c}(\mathbf{2}) \to \mathbb{R}\mathrm{c}(\mathbf{3})$:



Altogether, there is a well-defined right-connected double category $\Gamma(\mathbb{D})$.

Another canonical forgetful double functor

Given a double category $\mathbb{D} = \langle \mathcal{D}_0, \mathcal{D}_1 \rangle$ there is a double functor:

$$\Gamma(\mathbb{D}) \xrightarrow{\quad V \quad} \mathbb{D}$$

$$\begin{array}{ccc}
\Gamma_1(\mathbb{D}) & \xrightarrow{\quad V_1 \quad} & \mathcal{D}_1 \\
\mathrm{dom} \downarrow \;\; \uparrow \mathrm{id} \dashv \;\; \downarrow \mathrm{cod} & & \mathrm{dom} \downarrow \;\; \uparrow \mathrm{id} \;\; \downarrow \mathrm{cod} \\
\mathcal{D}_0 & = \!=\!=\!=\!= & \mathcal{D}_0
\end{array}$$

$$\begin{array}{ccc}
A & \xrightarrow{\;h\;} & C \\
{\scriptstyle (f, f', \alpha)} \big\downarrow & \theta & \big\downarrow {\scriptstyle (g, g', \alpha)} \\
B & \xrightarrow[\;k\;]{} & D
\end{array}
\quad \longmapsto \quad
\begin{array}{ccc}
A & \xrightarrow{\;h\;} & C \\
{\scriptstyle f} \big\downarrow & \theta & \big\downarrow {\scriptstyle g} \\
B & \xrightarrow[\;k\;]{} & D
\end{array}$$

The right-connected completion as a right 2-adjoint

Let $\mathbb{\Gamma}\colon \mathcal{D}\mathrm{BL} \to \mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL}$ denote the 2-functor which assigns a double category $\mathbb{D}$ to its right-connected completion $\mathbb{\Gamma}(\mathbb{D})$.

Main theorem

There is a 2-adjunction:

$$\mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL} \underset{\mathbb{\Gamma}}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathcal{D}\mathrm{BL}$$

The double functor $V\colon \mathbb{\Gamma}(\mathbb{D}) \to \mathbb{D}$ is the component at $\mathbb{D}$ of the counit of the 2-adjunction, while the unit is an isomorphism.

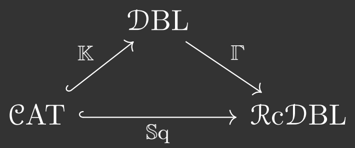
Example: The codiscrete double category

Let $\mathbb{K}(\mathcal{C})$ denote the *codiscrete double category* of a category $\mathcal{C}$ whose:

- objects and horizontal morphisms come from $\mathcal{C}$;
- vertical morphisms and cells come from $\mathcal{C} \times \mathcal{C}$.

This defines a 2-functor $\mathbb{K} \colon \mathcal{C}\mathrm{AT} \to \mathcal{D}\mathrm{BL}$ which is right adjoint to $\mathrm{Obj} \colon \mathcal{D}\mathrm{BL} \to \mathcal{C}\mathrm{AT}$.

There is a commutative diagram of 2-functors:



The double functor $\mathbb{D} \to \mathbb{K}(\mathcal{D}_0)$ is sent to the double functor $U \colon \Gamma(\mathbb{D}) \to \mathbb{S}\mathrm{q}(\mathcal{D}_0)$.

Example: The vertical dual of $\mathbb{S}\mathrm{q}(\mathcal{C})$

Let $\mathbb{S}\mathrm{q}(\mathcal{C})^{\mathsf{v}}$ denote the *vertical dual* of $\mathbb{S}\mathrm{q}(\mathcal{C})$, obtained by swapping the functors $\mathrm{dom}, \mathrm{cod} \colon \mathcal{S}\mathrm{q}(\mathcal{C}) \to \mathcal{C}$ of the internal category corresponding to $\mathbb{S}\mathrm{q}(\mathcal{C})$.

$$
\begin{array}{ccc}
A & \xrightarrow{\;h\;} & C \\
{\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\
B & \xrightarrow[\;k\;]{} & D
\end{array}
$$

There is an isomorphism $\Gamma\big(\mathbb{S}\mathrm{q}(\mathcal{C})^{\mathsf{v}}\big) \cong \mathbb{S}\mathrm{Epi}(\mathcal{C})$. A vertical morphism in $\Gamma\big(\mathbb{S}\mathrm{q}(\mathcal{C})^{\mathsf{v}}\big)$ is precisely a split epimorphism.

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & B \\
{\scriptstyle \varphi}\uparrow & & \uparrow{\scriptstyle 1_B} \\
B & \xrightarrow[\;1_B\;]{} & B
\end{array}
$$

Example: The double category of pushout squares

Let $\mathbb{P}\mathrm{o}(\mathcal{C})$ denote the *double category of pushout squares* in a category $\mathcal{C}$, obtained by restricting the cells in $\mathbb{S}\mathrm{q}(\mathcal{C})$ to those which are pushouts.

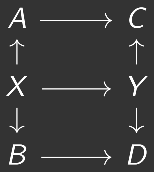
$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$$

A vertical morphism in $\Gamma\big(\mathbb{P}\mathrm{o}(\mathcal{C})\big)$ is an epimorphism.

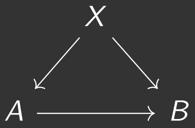
$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle e}\downarrow & \ulcorner & \downarrow{\scriptstyle 1_B} \\ B & \xrightarrow[1_B]{} & B \end{array}$$

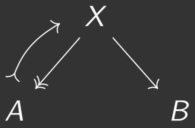
Example: The double category of spans

Let $\mathbb{S}\mathrm{pan}$ denote the double category of sets, functions, and spans of functions.



A vertical morphism in $\mathbb{\Gamma}(\mathbb{S}\mathrm{pan})$ is a composable pair of functions, while a vertical morphism in $\mathbb{\Gamma}(\mathbb{S}\mathrm{pan}^{\mathsf{h}})$ is a split multi-valued function.





Example: Double categories from 2-categories

Let $\mathbb{Q}(\mathcal{K})$ denote the *double category of quintets* of a 2-category $\mathcal{K}$, whose cells are:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\big\downarrow & \overset{\alpha}{\Longrightarrow} & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$$

A vertical morphism in $\mathbb{\Gamma}(\mathbb{Q}(\mathcal{K}))$ is simply a 2-cell $\alpha \colon f \Rightarrow g$ in $\mathcal{K}$.

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle f}\big\downarrow & \overset{\alpha}{\Longrightarrow} & \big\downarrow{\scriptstyle 1_B} \\ B & \xrightarrow[1_B]{} & B \end{array}$$

Vertical morphisms in $\mathbb{\Gamma}(\mathbb{H}(\mathcal{K}))$ and $\mathbb{\Gamma}(\mathbb{V}(\mathcal{K}))$ are pointed and copointed endomorphisms, respectively.

Example: The double category of adjunctions

Let $\mathbb{A}\mathrm{dj}(\mathcal{K})$ denote the double category of adjunctions in a 2-category $\mathcal{K}$.

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ r_1 \uparrow \vdash \downarrow \ell_1 & & r_2 \uparrow \vdash \downarrow \ell_2 \\ B & \xrightarrow[k]{} & D \end{array} \qquad k \circ \ell_1 = \ell_2 \circ h \qquad h \circ r_1 = r_2 \circ k$$

A vertical morphism in $\mathbb{\Gamma}(\mathbb{A}\mathrm{dj}(\mathcal{K}))$ is a 1-cell $\ell \colon A \to B$ equipped with a right-adjoint right-inverse (*rari*) $r \colon B \to A$. If $\mathcal{K}$ has cocomma objects, then $\mathbb{\Gamma}(\mathbb{A}\mathrm{dj}(\mathcal{K})) \cong R\text{-}\mathbb{A}\mathrm{lg}$ for an AWFS.

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ r \uparrow \vdash \downarrow \ell & & 1_B \uparrow \vdash \downarrow 1_B \\ B & \xrightarrow[1_B]{} & B \end{array} \qquad \ell \circ r = 1_B$$

Digression: Cofunctors

A cofunctor $\varphi \colon A \nrightarrow B$ consists of an object assignment $\varphi_0 \colon A_0 \to B_0$ together with a lifting operation

$$
\begin{array}{ccccc}
A & & a & \xrightarrow{\varphi(a,u)} & p(a,u) \\
{\scriptstyle\varphi}\big\downarrow\!\!\!\!\!\!\;+ & & \vdots & & \vdots \\
B & & \varphi_0 a & \xrightarrow{\quad u \quad} & b
\end{array}
$$

where $p(a,u) := \mathrm{cod}(\varphi(a,u))$, satisfying the following three axioms:

1. $\varphi_0 p(a,u) = \mathrm{cod}(u)$,
2. $\varphi(a, 1_{\varphi_0 a}) = 1_a$,
3. $\varphi(a, v \circ u) = \varphi(p(a,u), v) \circ \varphi(a,u)$.

Every delta lens $(f, \varphi) \colon A \to B$ has an underlying cofunctor $\varphi \colon A \nrightarrow B$.

Example: The double category of cofunctors

Let $\mathbb{C}\mathrm{of}$ denote the *double category of cofunctors* whose:

- objects are categories;
- horizontal morphisms are functors;
- vertical morphisms are cofunctors;
- cells are compatible squares

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\gamma} \\ B & \xrightarrow[k]{} & D \end{array}$$

s.t. $\gamma_0 h(a) = k\varphi_0(a)$ and $h\varphi(a, u) = \gamma(ha, ku)$ for all $(a \in A, u \colon \varphi_0 a \to b \in B)$.

Then $\Gamma(\mathbb{C}\mathrm{of}) \cong \mathbb{L}\mathrm{ens}$, and a delta lens is compatible functor and cofunctor pair.

Summary of the talk

- We constructed the right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ of a double category $\mathbb{D}$, which equips each vertical morphism with the structure of a horizontal morphism.

$$\mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL} \underset{\mathbb{\Gamma}}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathcal{D}\mathrm{BL}$$

- In particular examples, we saw that the right-connected completion is isomorphic to a double category arising from an AWFS.

$$\mathbb{\Gamma}(\mathbb{S}\mathrm{q}(\mathcal{C})^{\mathsf{v}}) \cong \mathbb{S}\mathrm{Epi}(\mathcal{C}) \qquad \mathbb{\Gamma}(\mathbb{A}\mathrm{dj}(\mathcal{K})) \cong \mathbb{L}\mathrm{ali}(\mathcal{K}) \qquad \mathbb{\Gamma}(\mathbb{C}\mathrm{of}) \cong \mathbb{L}\mathrm{ens}$$

- Further work is needed to determine conditions on $\mathbb{D} = \langle \mathcal{D}_0, \mathcal{D}_1 \rangle$ such that the functor $U_1 \colon \Gamma_1(\mathbb{D}) \to \mathcal{S}\mathrm{q}(\mathcal{D}_0)$ is strictly monadic, and corresponds to an AWFS.

Bonus: The canonical span of forgetful double fucntors

Given a double category $\mathbb{D} = \langle \mathcal{D}_0, \mathcal{D}_1 \rangle$ there is a span of double functor:

$$\mathbb{S}\mathrm{q}(\mathcal{D}_0) \xleftarrow{\;U\;} \mathbb{\Gamma}(\mathbb{D}) \xrightarrow{\;V\;} \mathbb{D}$$

$$\begin{array}{ccccc}
\mathcal{S}\mathrm{q}(\mathcal{D}_0) & \xleftarrow{\;U_1\;} & \Gamma_1(\mathbb{D}) & \xrightarrow{\;V_1\;} & \mathcal{D}_1 \\
\text{dom}\downarrow \;\; \uparrow \text{id} \vdash \;\; \downarrow \text{cod} & & \text{dom}\downarrow \;\; \uparrow \text{id} \vdash \;\; \downarrow \text{cod} & & \text{dom}\downarrow \;\; \uparrow \text{id} \;\; \downarrow \text{cod} \\
\mathcal{D}_0 & = & \mathcal{D}_0 & = & \mathcal{D}_0
\end{array}$$

$$\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\
B & \xrightarrow[k]{} & D
\end{array}
\quad \longmapsfrom \quad
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle (f, f', \alpha)}\downarrow & \theta & \downarrow{\scriptstyle (g, g', \alpha)} \\
B & \xrightarrow[k]{} & D
\end{array}
\quad \longmapsto \quad
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle f}\downarrow & \theta & \downarrow{\scriptstyle g} \\
B & \xrightarrow[k]{} & D
\end{array}$$

Bonus: Reinterpreting this span via a cospan

Using the canonical cospan of double functors $\mathbb{H}(\mathcal{C}) \to \mathbb{R}\mathrm{c}(\mathcal{C}) \leftarrow \mathbb{V}(\mathcal{C})$ we have:

$$\mathbb{S}\mathrm{q}(\mathcal{D}_0) \xleftarrow{\quad U \quad} \Gamma(\mathbb{D}) \xrightarrow{\quad V \quad} \mathbb{D}$$

$$\begin{array}{ccccc}
\mathcal{D}\mathrm{BL}\big(\mathbb{H}(\mathbf{2}),\mathbb{D}\big) & \xleftarrow{\quad U_1 \quad} & \mathcal{D}\mathrm{BL}\big(\mathbb{R}\mathrm{c}(\mathbf{2}),\mathbb{D}\big) & \xrightarrow{\quad V_1 \quad} & \mathcal{D}\mathrm{BL}\big(\mathbb{V}(\mathbf{2}),\mathbb{D}\big) \\
\mathrm{dom}\downarrow \;\; \uparrow \mathrm{id} \dashv \;\; \downarrow \mathrm{cod} & & \mathrm{dom}\downarrow \;\; \uparrow \mathrm{id} \dashv \;\; \downarrow \mathrm{cod} & & \mathrm{dom}\downarrow \;\; \uparrow \mathrm{id} \;\; \downarrow \mathrm{cod} \\
\mathcal{D}\mathrm{BL}\big(\mathbb{H}(\mathbf{1}),\mathbb{D}\big) & = & \mathcal{D}\mathrm{BL}\big(\mathbb{R}\mathrm{c}(\mathbf{1}),\mathbb{D}\big) & = & \mathcal{D}\mathrm{BL}\big(\mathbb{V}(\mathbf{1}),\mathbb{D}\big)
\end{array}$$

- Under what conditions does U_1 have a left adjoint?
- Answer: When left Kan extensions of $\mathbb{V}(\mathbf{2}) \to \mathbb{D}$ along $\mathbb{V}(\mathbf{2}) \to \mathbb{R}\mathrm{c}(\mathbf{2})$ in exist.
- Under what conditions is V_1 comonadic?
- Answer: If and only if each fibre $\mathrm{cod}^{-1}\{B\}$ of the functor $\mathrm{cod}\colon \mathcal{D}_1 \to \mathcal{D}_0$ admits products with the vertical identity morphism $\mathrm{id}_B \colon B \nrightarrow B$.

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