An Introduction to Enriched Cofunctors

arXiv:2209.01144

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joint work with

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Theoretical Cosynus Seminar

Inria Saclay / LIX, 6 October 2022

OVERVIEW OF THE TALK

(1) Examples of enriched cofunctors

- Weighted categories, Lawvere metric spaces, algebroids/linear categories.

(2) Defining enriched cofunctors

- In a distributive monoidal category
- As certain spans of enriched functors in an extensive category

(3) Duality and compatibility

- In what sense are functors & cofunctors dual?
- Double categories of enriched functors & cofunctors.

WHAT IS A COFUNCTOR?

A cofunctor $\varphi : A \nrightarrow B$ consists of

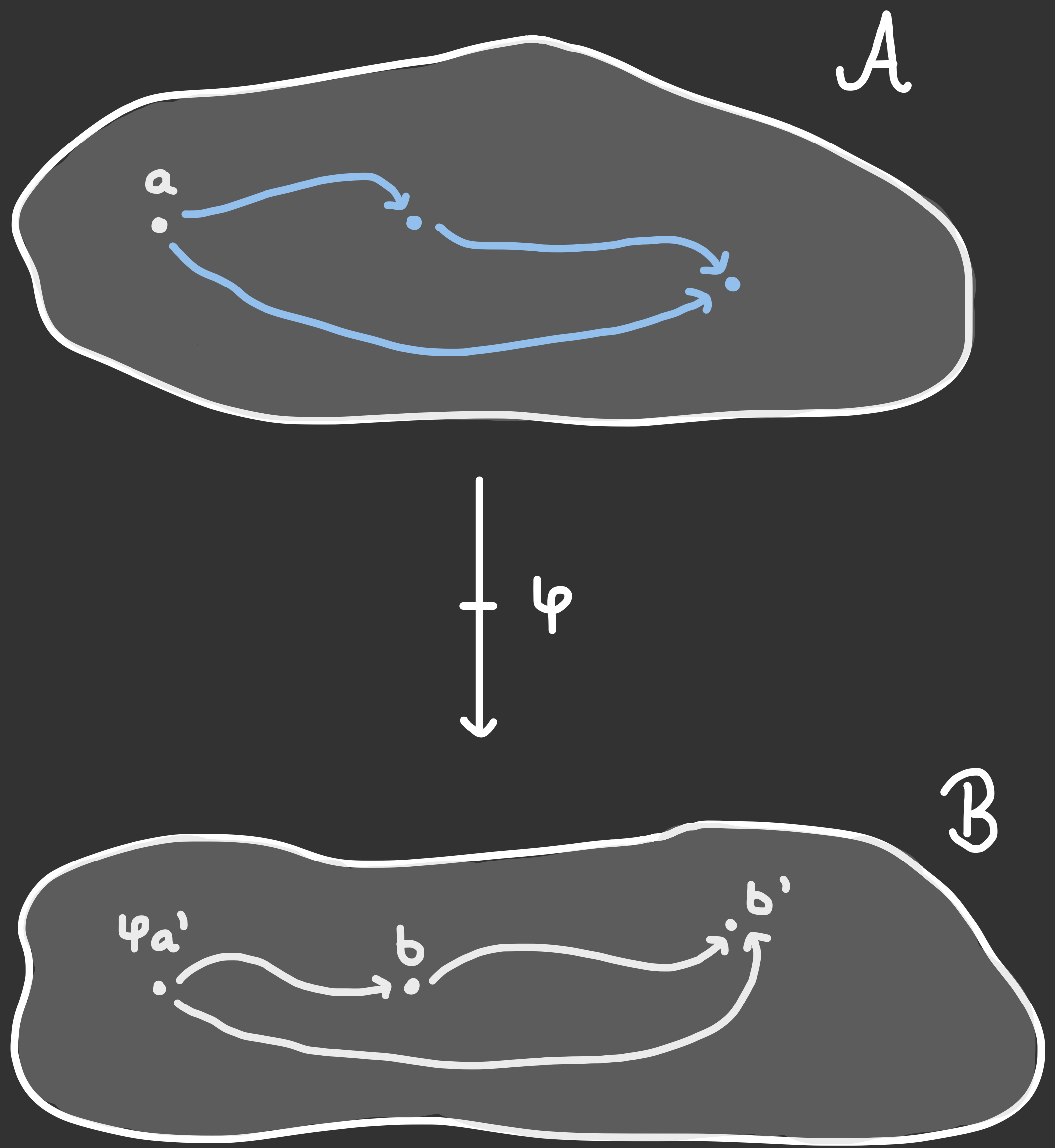
an object assignment

$$\varphi : \mathrm{obj}(A) \longrightarrow \mathrm{obj}(B)$$

and a lifting operation

$$\begin{tikzcd} A \arrow[d, "\varphi"', "|" marking] & a \arrow[r, "\varphi^{\#}(a,u)"] \arrow[d, dotted, no head] & a' \arrow[d, dotted, no head] \\ B & \varphi a \arrow[r, "u"] & b = \varphi a' \end{tikzcd}$$

which preserves identities & composition.



COFUNCTORS BETWEEN WEIGHTED CATEGORIES

A weighted category is a category enriched in weighted sets.

- Each morphism has a weight
  $$|u| \in [0, \infty]$$
- The following axioms hold:
  $$|1_x| = 0$$
  identities have weight 0
  $$|u \circ v| \leqslant |u| + |v|$$
  triangle inequality

A weighted functor f satisfies
$$|f(u)| \leqslant |u|.$$

A weighted cofunctor φ satisfies
$$|\varphi^{\#}(a, u)| \leqslant |u|.$$

Introduced by Perrone (2021) to capture the notion of lifting transport plans between probability measures on standard Borel spaces while preserving their cost.

COFUNCTORS BETWEEN LAWVERE METRIC SPACES

A **Lawvere metric space** is a category enriched in $([0,\infty], \geqslant)$ under addition.

- For each pair of objects, a **distance**

$$d(x,y) \in [0,\infty]$$

- The following axioms hold:

$$d(x,x) = 0$$

reflexivity

$$d(x,y) + d(y,z) \geqslant d(x,z)$$

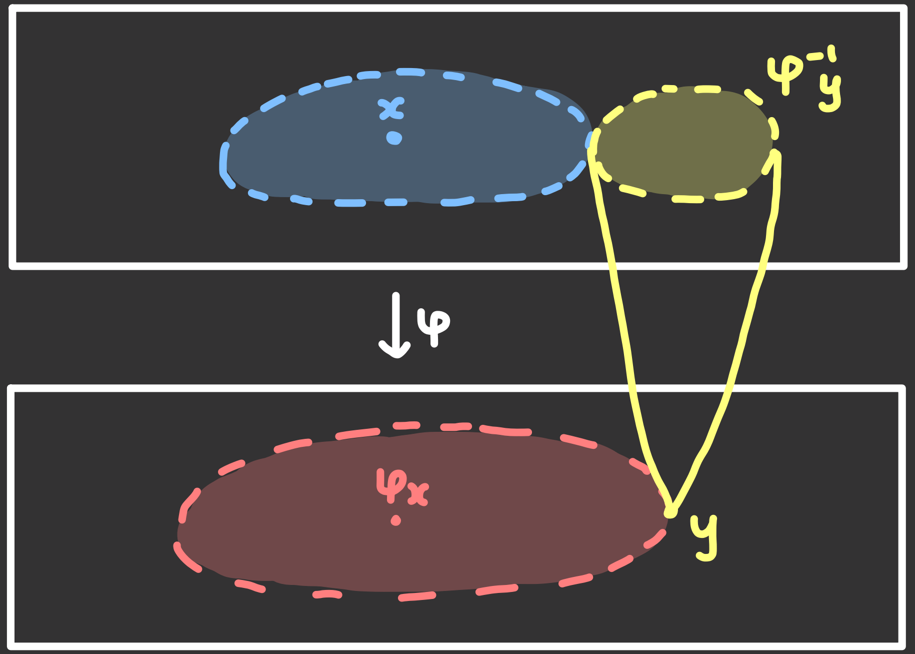
triangle inequality

Functors ⇝ 1-Lipschitz functions

$$d(x,y) \geqslant d(fx, fy)$$

Cofunctors ⇝ weak submetries

$$d(\varphi x, y) \geqslant \inf\{ d(x,z) \mid z \in \varphi^{-1}y \}$$



COFUNCTORS BETWEEN LINEAR CATEGORIES

A **linear category** (or **algebroid**) is a category enriched in vector spaces.

- For each pair of objects, the hom-set has a **vector space** structure:

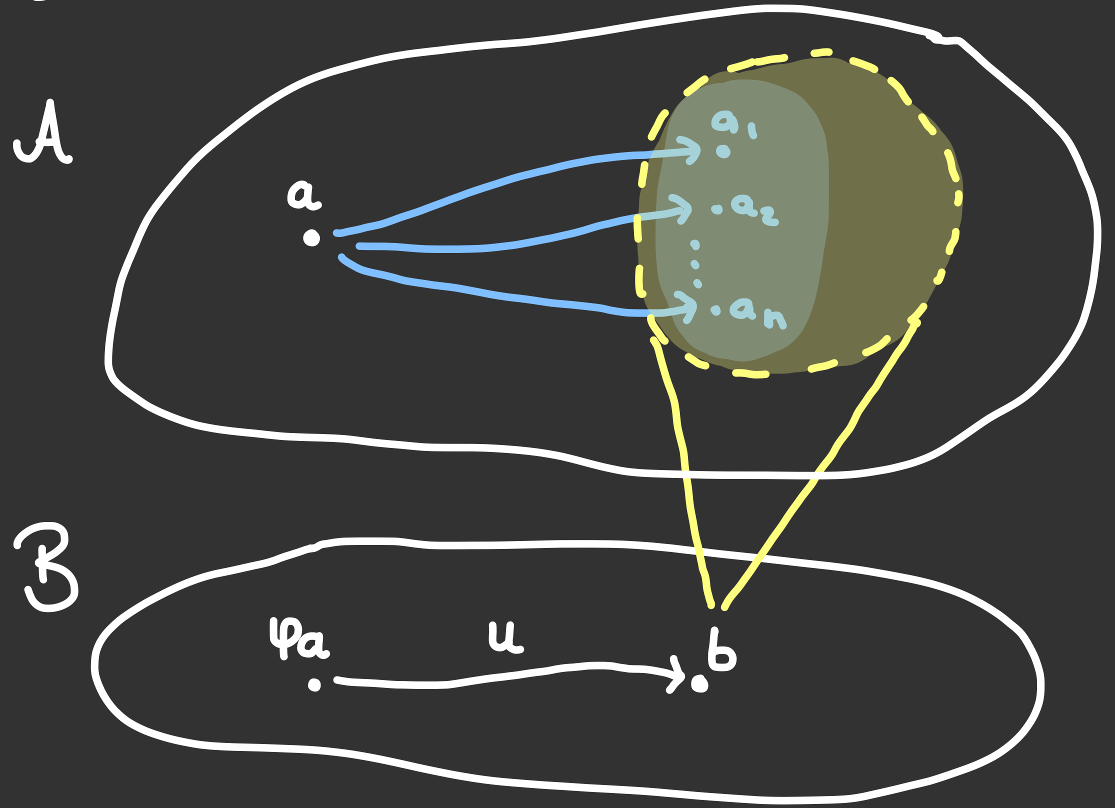
$$A(a, a') \in \mathrm{Vec}_k$$

- Composition is **bilinear**:

$$A(a,a') \times A(a',a'') \xrightarrow{\text{comp}} A(a,a'')$$

A linear category with a single object is an **algebra** over a field k.

A **linear cofunctor** $\varphi : A \nrightarrow B$ chooses a family of lifts indexed by a **finite subset** of the fibre:



WHAT IS AN ENRICHED COFUNCTOR?

Let $\mathcal{V}$ be a distributive monoidal cat.

An enriched cofunctor $\Phi : \mathcal{A} \nrightarrow \mathcal{B}$ consists of an object assignment $\Phi : \mathrm{obj}(\mathcal{A}) \longrightarrow \mathrm{obj}(\mathcal{B})$ and a family

$$\Phi_{a,b} : \mathcal{B}(\Phi a, b) \longrightarrow \sum_{x \in X} \mathcal{A}(a,x)$$

of morphisms in $\mathcal{V}$, where $X = \Phi^{-1}\{b\}$, such that the following diagrams commute.

respects identities

$$\begin{tikzcd}
I \arrow[r, "\eta_{\Phi a}"] \arrow[d, "\eta_a"'] & \mathcal{B}(\Phi a, \Phi a) \arrow[d, "\Phi_{a,\Phi a}"] \\
\mathcal{A}(a,a) \arrow[r, "\iota_a"'] & \sum_{x \in X} \mathcal{A}(a,x)
\end{tikzcd}
\qquad X = \Phi^{-1}\{\Phi a\}$$

$$\begin{tikzcd}
\mathcal{B}(\Phi a, b) \otimes \mathcal{B}(b, b') \arrow[r, "\mu_{\Phi a, b, b'}"] \arrow[d, "\Phi_{a,b} \otimes \mathrm{id}"'] & \mathcal{B}(\Phi a, b') \arrow[dddd, "\Phi_{a,b'}"'] \\
\Big(\sum_{x \in X} \mathcal{A}(a,x)\Big) \otimes \mathcal{B}(b,b') \arrow[d, "\cong"'] & \\
\sum_{x \in X} \mathcal{A}(a,x) \otimes \mathcal{B}(\Phi x, b') \arrow[d, "\sum \mathrm{id} \otimes \Phi_{x,b'}"'] & \\
\sum_{x \in X} \mathcal{A}(a,x) \otimes \Big(\sum_{y \in Y} \mathcal{A}(x,y)\Big) \arrow[d, "\cong"'] & \\
\sum_{y \in Y} \sum_{x \in X} \mathcal{A}(a,x) \otimes \mathcal{A}(x,y) \arrow[r, "\sum [\mu_{a,x,y}]"'] & \sum_{y \in Y} \mathcal{A}(a,y)
\end{tikzcd}
\qquad X = \Phi^{-1}\{b\} \quad Y = \Phi^{-1}\{b'\}$$

respects composition

ENRICHED FUNCTORS VS. ENRICHED COFUNCTORS

	object assignment	morphism assignment
functor $F : A \longrightarrow B$	$F : \mathrm{obj}(A) \longrightarrow \mathrm{obj}(B)$	$[F_{a,x}] : \sum_{x \in F^{-1}\{b\}} A(a,x) \longrightarrow B(Fa,b)$
cofunctor $\Phi : A \nrightarrow B$	$\Phi : \mathrm{obj}(A) \longrightarrow \mathrm{obj}(B)$	$\Phi_{a,b} : B(\Phi a, b) \longrightarrow \sum_{x \in \Phi^{-1}\{b\}} A(a,x)$

ENRICHED COFUNCTORS AS SPANS

An enriched functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is:

- bijective-on-objects if
$$F : \mathrm{obj}(\mathcal{A}) \longrightarrow \mathrm{obj}(\mathcal{B})$$
is invertible.

- an elevator* if
$$[F_{a,x}] : \sum_{x \in F^{-1}\{b\}} \mathcal{A}(a,x) \longrightarrow \mathcal{B}(Fa,b)$$
is invertible.

Proposition: If $\mathcal{V}$ is extensive, then enriched cofunctors are equivalent to spans of enriched functors of the form:

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "\text{b.o.o.}"'] \arrow[dr, "\text{elevator}"] & \\ \mathcal{A} & & \mathcal{B} \end{tikzcd}$$

COMPATIBLE SQUARES & ENRICHED LENSES

A square of enriched functors & cofunctors

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "\Phi"', "+" marking] & \mathcal{C} \arrow[d, "\Psi", "+" marking] \\ \mathcal{B} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd}$$

is called **compatible** if $G\Phi a = \Psi F a$ and

$$\begin{tikzcd} \mathcal{B}(\Phi a, b) \arrow[r, "G_{\Phi a, b}"] \arrow[d, "\Phi_{a,b}"'] & \mathcal{D}(\Psi F a, G b) \arrow[d, "\Psi_{Fa, Gb}"] \\ \sum_{x \in \Phi^{-1}\{b\}} \mathcal{A}(a, x) \arrow[r, "{[\iota_{Fx} \circ F_{a,x}]}"'] & \sum_{y \in \Psi^{-1}\{Gb\}} \mathcal{C}(Fa, y) \end{tikzcd}$$

An **enriched lens** $(F, \Phi) : \mathcal{A} \rightleftarrows \mathcal{B}$ is a compatible square of the form:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "\Phi"', "+" marking] & \mathcal{B} \arrow[d, "1_{\mathcal{B}}", "+" marking] \\ \mathcal{B} \arrow[r, "1_{\mathcal{B}}"'] & \mathcal{B} \end{tikzcd}$$

We have that:

$$\begin{tikzcd} & \sum_{x \in \Phi^{-1}\{b\}} \mathcal{A}(a, x) \arrow[dr, "{[F_{a,x}]}"] & \\ \mathcal{B}(\Phi a, b) \arrow[ur, "\Phi_{a,b}"] \arrow[rr, "\mathrm{id}"'] & & \mathcal{B}(\Phi a, b) \end{tikzcd}$$

ENRICHED NATURAL TRANSFORMATIONS

An **enriched natural transformation**

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\Downarrow \tau"] & C \arrow[d, "\Psi"] \\ B \arrow[r, "G"'] & D \end{tikzcd}$$

consists of a family of morphisms in $\mathcal{V}$

$$I \xrightarrow{\tau_a} \sum_{x \in X} \mathcal{C}(Fa, x)$$

where $X = \Psi^{-1}\{G\Phi a\}$, such that the following diagram commutes.

encodes naturality

$$\begin{tikzcd}[column sep=small] I \otimes \mathcal{B}(\Phi a, b) \arrow[d, "\tau_a \otimes G_{\Phi a, b}"'] & \mathcal{B}(\Phi a, b) \arrow[l, "\cong"'] \arrow[r, "\cong"] & \mathcal{B}(\Phi a, b) \otimes I \arrow[d, "\Phi_{a,b} \otimes \mathrm{id}"] \\ \Big(\sum_{x \in X} \mathcal{C}(Fa, x)\Big) \otimes \mathcal{D}(G\Phi a, Gb) \arrow[d, "\cong"'] & & \Big(\sum_{y \in Y} \mathcal{A}(a, y)\Big) \otimes I \arrow[d, "\cong"] \\ \sum_{x \in X} \mathcal{C}(Fa, x) \otimes \mathcal{D}(\Psi x, Gb) \arrow[d, "\sum \mathrm{id} \otimes \Psi_{x, Gb}"'] & & \sum_{y \in Y} \mathcal{A}(a, y) \otimes I \arrow[d, "\sum F_{a,y} \otimes \tau_y"] \\ \sum_{x \in X} \mathcal{C}(Fa, x) \otimes \Big(\sum_{z \in Z} \mathcal{C}(x, z)\Big) \arrow[d, "\cong"'] & & \sum_{y \in Y} \mathcal{C}(Fa, Fy) \otimes \Big(\sum_{z \in Z} \mathcal{C}(Fy, z)\Big) \arrow[d, "\cong"] \\ \sum_{z \in Z} \sum_{x \in X} \mathcal{C}(Fa, x) \otimes \mathcal{C}(x, z) \arrow[r, "{\sum[\mu_{Fa,x,z}]}"'] & \sum_{z \in Z} \mathcal{C}(Fa, z) & \sum_{z \in Z} \sum_{y \in Y} \mathcal{C}(Fa, Fy) \otimes \mathcal{C}(Fy, z) \arrow[l, "{\sum[\mu_{Fa,Fy,z}]}"] \end{tikzcd}$$

$$X = \Psi^{-1}\{G\Phi a\} \qquad Y = \Phi^{-1}\{b\} \qquad Z = \Psi^{-1}\{Gb\}$$

These form cells in a **double category** $\mathcal{V}$-$\mathbb{C}\mathrm{of}$.

ENRICHED NATURAL TRANSFORMATIONS

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\tau"] & \mathcal{C} \arrow[d, "\Psi"] \\ \mathcal{B} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd}$$

$\mathcal{A}$:

$$\begin{tikzcd} a \arrow[r, "\Phi(a{,}u)"] & a' \end{tikzcd}$$

$\mathcal{B}$:

$$\begin{tikzcd} \Phi a \arrow[r, "u"] & b \arrow[d, equal] \\ & \Phi a' \end{tikzcd}$$

$\mathcal{C}$:

$$\begin{tikzcd} Fa \arrow[r, "F\Phi(a{,}u)"] \arrow[d, "\tau_a"'] & Fa' \arrow[d, "\tau_{a'}"] \\ c \arrow[r, "\Psi(c{,}Gu)"'] & c' \end{tikzcd}$$

$\mathcal{D}$:

$$\begin{tikzcd} G\Phi a \arrow[r, "Gu"] \arrow[d, equal] & G\Phi a' \arrow[d, equal] \\ \Psi c \arrow[ur, "Gu"'] & \Psi c' \end{tikzcd}$$

SUMMARY & FUTURE WORK

- Enriched cofunctors can be defined over distributive monoidal categories.
- Examples include weighted cofunctors and weak submetries.
- There is a double category $\mathcal{V}$-$\mathbb{C}\mathrm{of}$ of enriched functors and cofunctors.
- If $\mathcal{V}$ is extensive, then enriched cofunctors $\simeq$ certain spans.

- Show how enriched cofunctors arise as monad retromorphisms in the double category $\mathcal{V}$-$\mathbb{M}\mathrm{at}$.
- Develop a new perspective on labelled transition systems and bisimulation via quantale enrichment.
- Investigate possibility of an enriched cofunctor cat. $\mathcal{V}\text{-}\mathrm{Cat}^{*}(A,B)$.