Enriched Lenses

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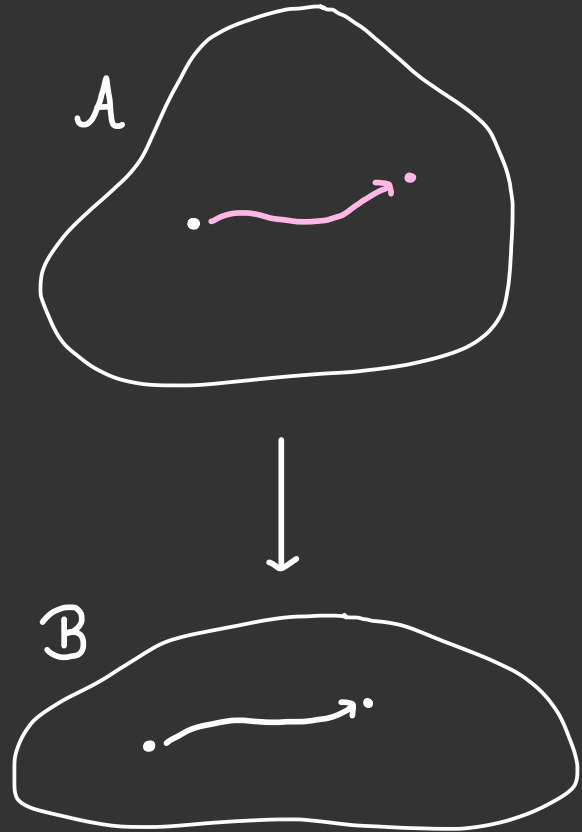
OVERVIEW & MOTIVATION

Lenses model bidirectional transformations:

- object assignment ⤳ consistency relation
- functor ⤳ forwards consistency restoration
- cofunctor ⤳ backwards consistency restoration

MOTIVATING QUESTION

What is the correct way to restore consistency if the systems are enriched?

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1. Introduce enriched cofunctors & lenses.
2. Share some examples, incl. weighted lenses.

ENRICHED FUNCTORS VS. COFUNCTORS

Assumption: enrichment in a **distributive monoidal category** $\mathcal{V}$

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enriched functor $F:\mathcal{A}\longrightarrow\mathcal{B}$

$$obj(\mathcal{A}) \xrightarrow{\ F\ } obj(\mathcal{B})$$

$$\sum_{x\in X}\mathcal{A}(a,x) \xrightarrow{\ [F_{a,x}]\ } \mathcal{B}(Fa,b)$$

$\underbrace{\phantom{\sum_{x\in X}}}$ $\boxed{X = F^{-1}\{b\}}$ + axioms

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enriched cofunctor $\langle F,\varphi\rangle:\mathcal{A}\longrightarrow\mathcal{B}$

$$obj(\mathcal{A}) \xrightarrow{\ F\ } obj(\mathcal{B})$$

$$\mathcal{B}(Fa,b) \xrightarrow{\ \varphi_{a,b}\ } \sum_{x\in X}\mathcal{A}(a,x)$$

$\boxed{X = F^{-1}\{b\}}$ + axioms

$\mathcal{V}=\omega Set$ weighted functor

$v\in\mathcal{A}(a,a')$ $|F_{a,a'}v|\leqslant|v|$
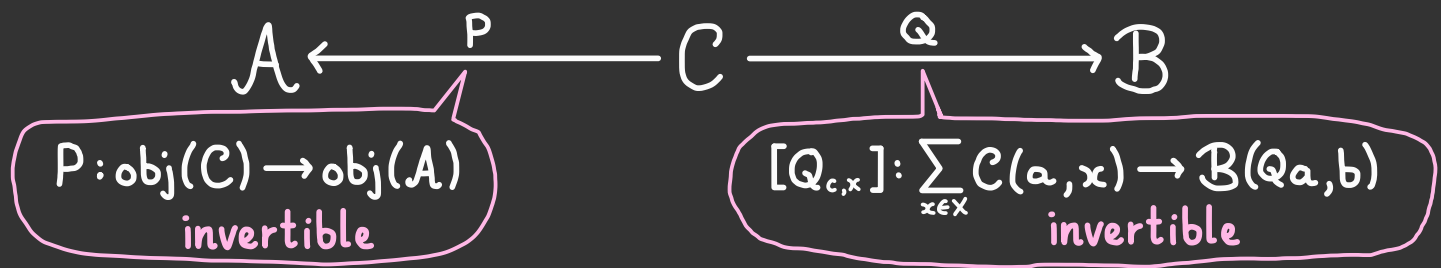

$\mathcal{V}=\omega Set$ weighted cofunctor

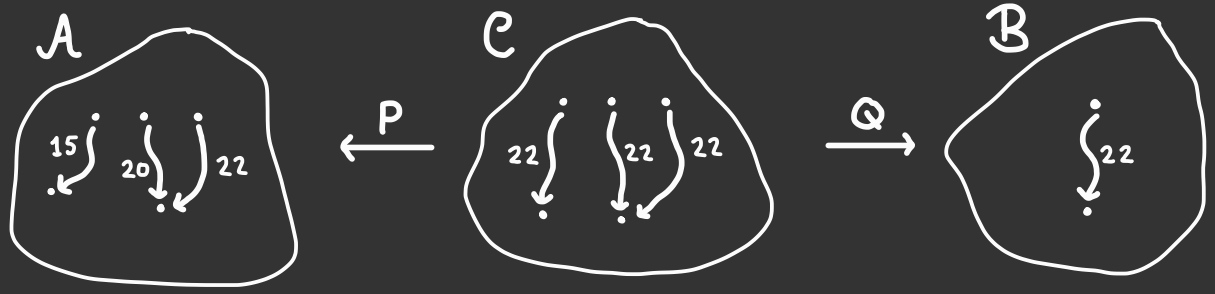$u\in\mathcal{B}(Fa,b)$ $|\varphi_{a,b}u|\leqslant|u|$

ENRICHED COFUNCTORS AS SPANS

Assumption: enrichment in a **distributive monoidal & extensive category** $\mathcal{V}$

enriched cofunctor $(F, \varphi): A \to B$ $\simeq$ span of **enriched functors**:

$$A \xleftarrow{P} C \xrightarrow{Q} B$$

$P: \mathrm{obj}(C) \to \mathrm{obj}(A)$
invertible

$[Q_{c,x}]: \sum_{x \in X} C(a,x) \to B(Qa, b)$
invertible

$\mathcal{V} = \omega\mathrm{Set}$
weighted cofunctor
$|\varphi_{a,b} u| \leqslant |u|$

ENRICHED LENSES

enriched functor $F: \mathcal{A} \longrightarrow \mathcal{B}$
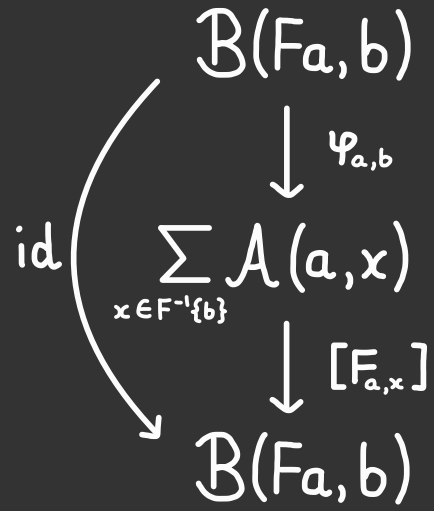
$$F_{a,a'}: \mathcal{A}(a,a') \longrightarrow \mathcal{B}(Fa, Fa')$$

$$F: obj(\mathcal{A}) \longrightarrow obj(\mathcal{B})$$

$$\varphi_{a,b}: \mathcal{B}(Fa, b) \longrightarrow \sum_{x \in F^{-1}\{b\}} \mathcal{A}(a,x)$$

enriched cofunctor $(F, \varphi): \mathcal{A} \longrightarrow \mathcal{B}$

enriched lens $\langle F, \varphi \rangle: \mathcal{A} \nrightarrow \mathcal{B}$

$$\mathcal{B}(Fa, b)$$

$$\text{id} \left(\begin{array}{c} \downarrow \varphi_{a,b} \\ \sum_{x \in F^{-1}\{b\}} \mathcal{A}(a,x) \\ \downarrow [F_{a,x}] \end{array} \right.$$

$$\mathcal{B}(Fa, b)$$

$\mathcal{V} = \omega Set$ weighted lens

$u \in \mathcal{B}(Fa, b)$ $\quad |\varphi_{a,b} u| = |u|$

EXAMPLES OF ENRICHED LENSES

$\mathcal{V}$	Coproduct	lifting map	notes
$(\mathrm{Set}, \times, 1)$	$\sqcup$	$\mathcal{B}(Fa,b) \xrightarrow{\varphi_{a,b}} \bigsqcup_{x \in F^{-1}\{b\}} \mathcal{A}(a,x)$	delta lens Diskin et al. 2011
$(\omega\mathrm{Set}, +, 0)$	$\sqcup$	same, s.t. $|\varphi u| = |u| \ \forall u$	weighted lens Perrone 2021
$([0,\infty], +, 0)$	inf	$d(Fa,b) \geqslant \inf_{x \in F^{-1}\{b\}} d(a,x)$	submetry is an example
$(\mathrm{Ab}, \otimes, \mathbb{Z})$	$\oplus$ biproduct	$\left\{ \mathcal{B}(Fa,b) \xrightarrow{\varphi_{a,b}} \mathcal{A}(a,x) \right\}_{x \in F^{-1}\{b\}}$	additive lens
$\mathrm{Fam}(\mathcal{V})$	$\sqcup$	similiar to first example	any monoidal category $\mathcal{V}$

WHY IS THIS NOTION "CORRECT"?

$$\mathbb{D} \rightsquigarrow \mathbb{M}nd_{ret}(\mathbb{D}) \rightsquigarrow \Gamma(\mathbb{M}nd_{ret}(\mathbb{D}))$$

double category with companions	monads, monad morphisms, & monad retromorphisms	right-connected completion $\rightsquigarrow$ lenses between monads
$\mathbb{M}at(\mathcal{V})$	enriched categories, functors, & cofunctors	enriched lenses
$\mathbb{S}pan(\mathcal{E})$	internal categories, functors, & cofunctors	internal lenses
$F\text{-}\mathbb{R}el$	topological spaces, continuous maps, & open maps	open continuous maps

SUMMARY & FUTURE WORK

| enriched lens | $\rightsquigarrow$ | vertical arrow in $\mathbb{\Gamma}(\mathbb{M}\text{nd}_{\text{ret}}(\mathbb{M}\text{at}(\mathcal{V})))$ |

$=$

enriched functor

$\mathcal{V} = \text{Set}$ $\mathcal{V} = [0, \infty]$
$\mathcal{V} = \omega\text{Set}$ $\mathcal{V} = \text{Ab}$

$+$

| enriched cofunctor | $\simeq$ | span of enriched functors |

$\mathcal{V}$ distributive monoidal

$\mathcal{V}$ extensive

More examples!
- $\mathcal{V} = \text{Poly} \rightsquigarrow$ collectives
- $\mathcal{V} = \mathcal{P}M$, M monoid, $\mathcal{P}$ powerset

More theory!
- Can we better understand enriched split opfibrations using enriched lenses?
- Grothendieck construction for weighted lenses? Enriched?
- 2-cells of enriched cofunctors.