INVESTIGATING LENSES BETWEEN PREORDERED SETS

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Proofs and Algorithms seminar
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OVERVIEW OF THE TALK

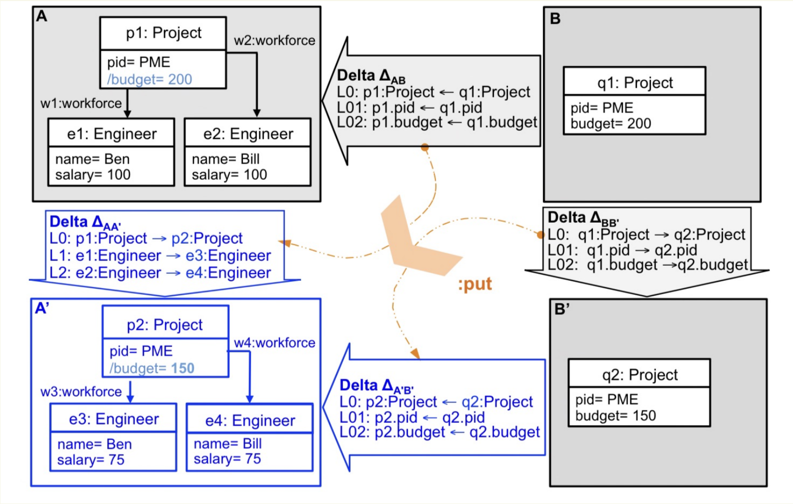
1. Motivation for lenses

2. Lenses between sets

3. Lenses between preordered sets & investigate their properties:
   - How to compose & decompose
   - How to take intersections & glue
   - How to construct free lenses

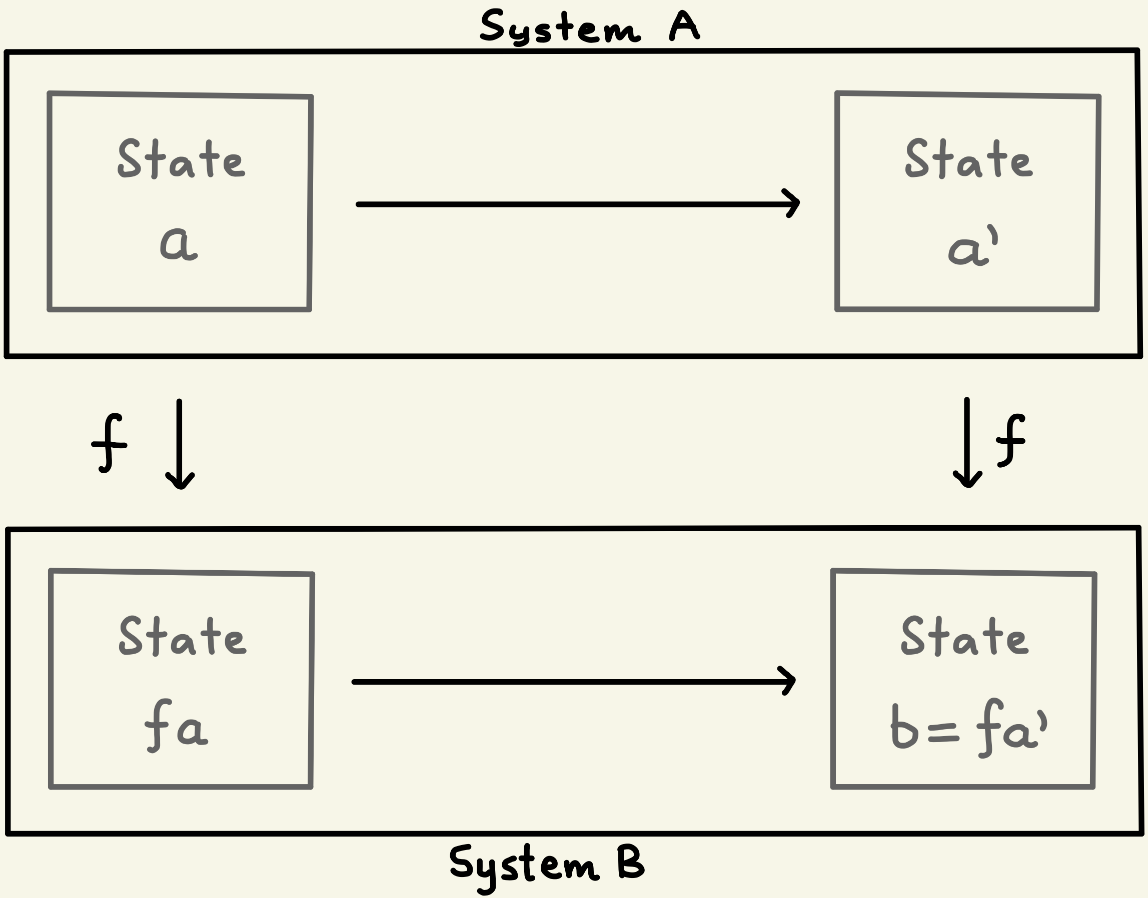
4. Lenses between enriched structures

Example from model-driven engineering:



Diskin, Eramo, Pierantonio, and Czarnecki (2016).

MOTIVATION



Summary:

- This is a simple model of a system
- We would like control over which updates are permitted
- Idea: Replace sets with preorders!

LENSES BETWEEN SETS

A lens $(f,p): A \longrightarrow B$ consists of a pair of functions,

$$f: A \longrightarrow B \qquad p: A \times B \longrightarrow A$$

which satisfy the axioms:

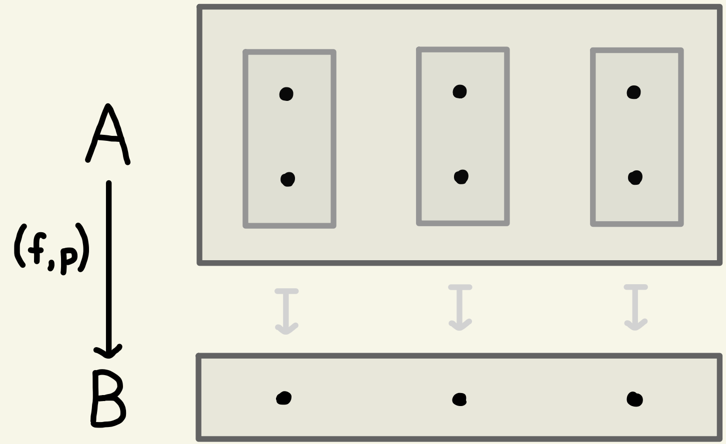
(1) $$fp(a,b) = b$$

(2) $$p(a, fa) = a$$

(3) $$p(p(a,b), b') = p(a,b')$$

Example: $(f,p): \mathbb{R}^2 \longrightarrow \mathbb{R}$ s.t.

$$f(x,y) = y \qquad p(x,y,z) = (x,z)$$



Issues with lenses between sets:

- Not always a realistic model
- Only examples are product projections
- Mathematically poor properties

LENSES BETWEEN PREORDERED SETS

A preordered set $(A, \leqslant)$ is a set A equipped with a relation $\leqslant$ which satisfies the axioms:

(Reflexivity) $\quad x \leqslant x$

(Transitivity) $\quad$ if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$

A function $f : A \longrightarrow B$ between preordered sets is order preserving if

$$x \leqslant y \quad \text{implies} \quad fx \leqslant fy.$$

A lens $(f, p) : (A, \leqslant) \longrightarrow (B, \leqslant)$ consists of an order-preserving function $f : A \longrightarrow B$ and a partial function,

$$fa \leqslant b \longmapsto a \leqslant p(a, b)$$

which satisfy the axioms:

(1) $\quad fp(a, b) = b$

(2) $\quad p(a, fa) = a$

(3) $\quad p(p(a, b), b') = p(a, b')$

for all $fa \leqslant b \leqslant b'$

BASIC EXAMPLES

Addition is an order-preserving function:

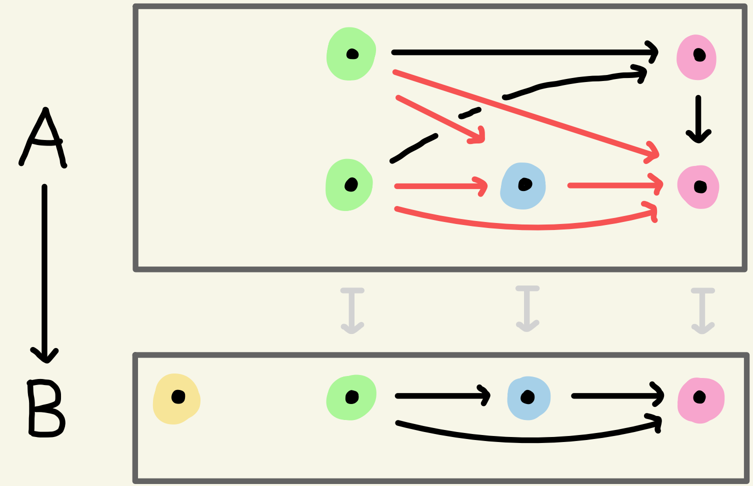
$$+ : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto x+y$$

Three possible lens structures include:

$$x+y \leq z \longmapsto (z-y,\, y)$$
$$x+y \leq z \longmapsto (x,\, z-x)$$
$$x+y \leq z \longmapsto \left(\frac{z+x-y}{2},\, \frac{z+y-x}{2}\right)$$

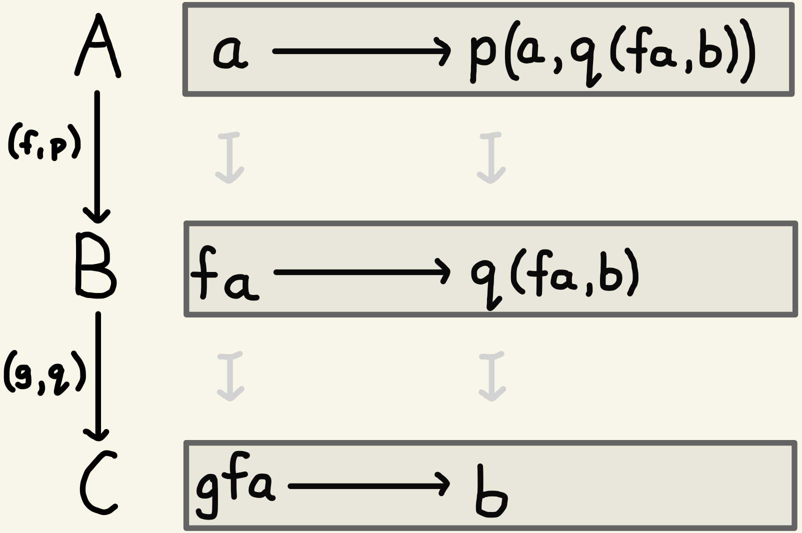
Every lens between sets is an example under the relation $x \leq y$ for all (x,y).

Represent a preordered set by a graph:



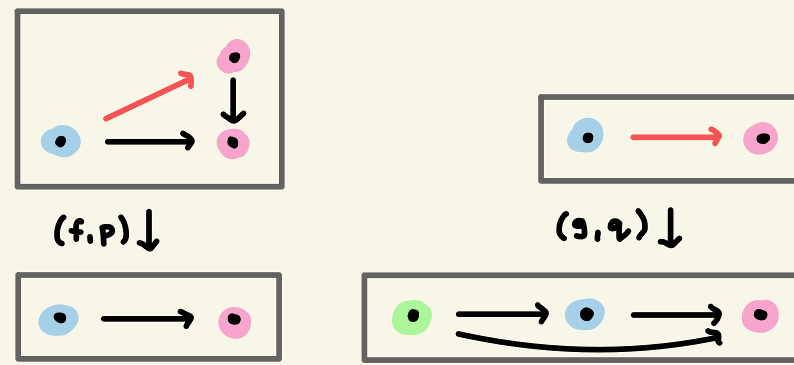
A lens lifts an arrow in the target graph B to an arrow in the source graph A.

HOW TO COMPOSE, SEQUENTIALLY?

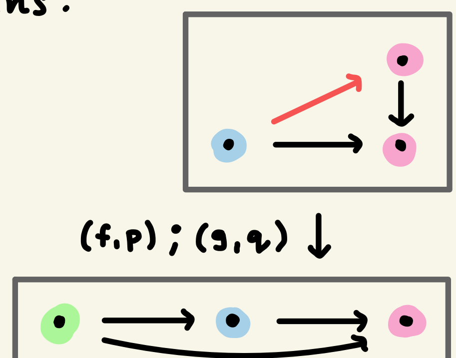


Composition of lenses is unital and associative.

Example: The composite of



is the lens:

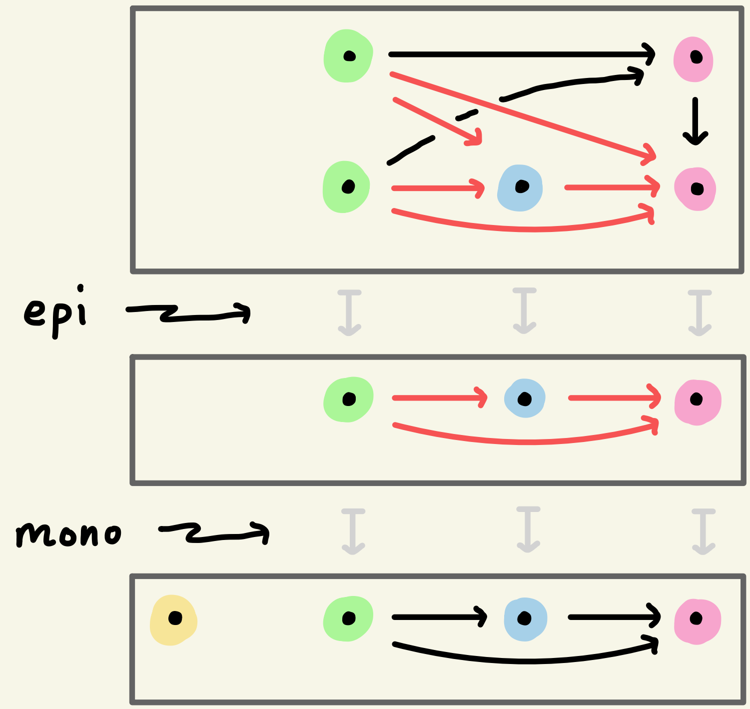


HOW TO DECOMPOSE, SEQUENTIALLY?

Every function decomposes into a surjection followed by an injection.

A lens is a (epi / mono) if its underlying function is a (surjection / injection).

Every lens decomposes into an epi followed by a mono – this is a factorisation system on lenses.



PRODUCTS & COPRODUCTS

There are at least two ways of combining lenses in parallel.

Consider the following two simple examples:

$$\boxed{x \longrightarrow y} \;\xrightarrow{f}\; \boxed{x \longrightarrow y} \qquad\qquad \boxed{\begin{array}{c} a \\ \downarrow \\ b \end{array}} \;\xrightarrow{g}\; \boxed{z}$$

coproduct

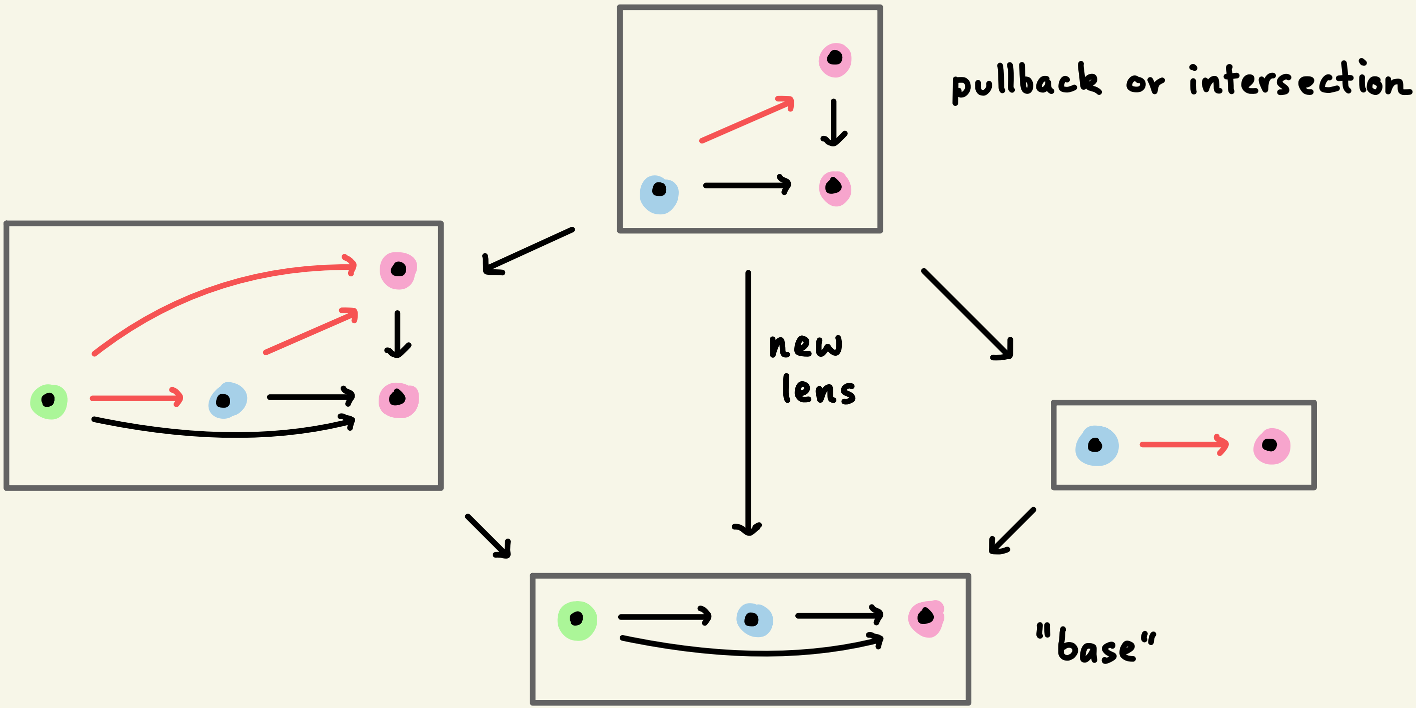
$$\boxed{\begin{tikzcd} & & a \arrow[d] \\ x \arrow[r] & y & b \end{tikzcd}} \;\xrightarrow{f \oplus g}\; \boxed{x \longrightarrow y \qquad z}$$

monoidal product

$$\boxed{\begin{tikzcd} (x,a) \arrow[r] \arrow[d] \arrow[dr] & (y,a) \arrow[d] \\ (x,b) \arrow[r] & (y,b) \end{tikzcd}} \;\xrightarrow{f \otimes g}\; \boxed{(x,z) \longrightarrow (y,z)}$$

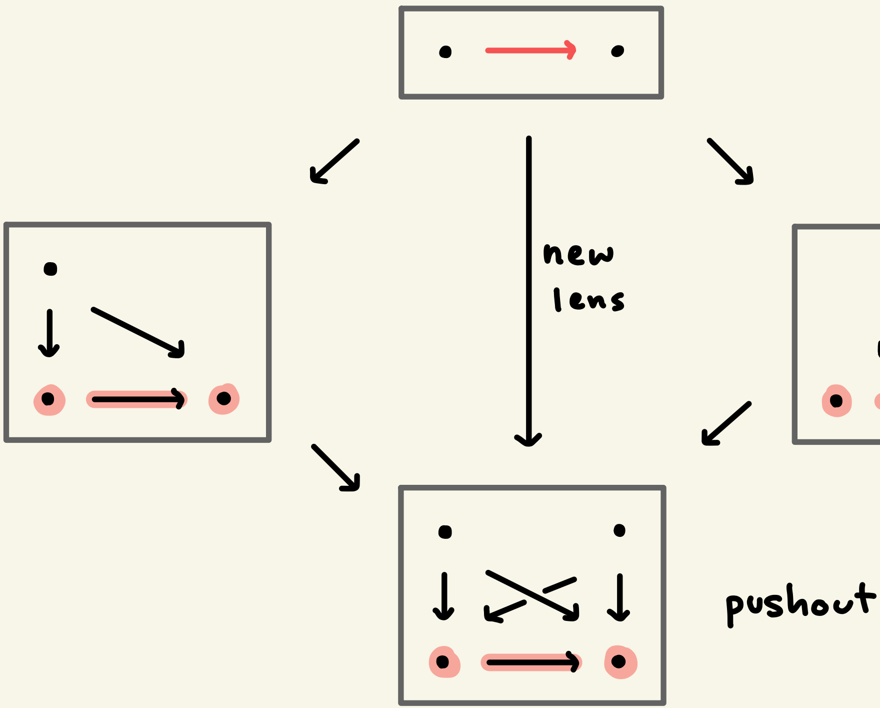
PULLBACKS

Pullbacks of lenses are akin to taking an intersection, or taking a product over some "base" preorder.



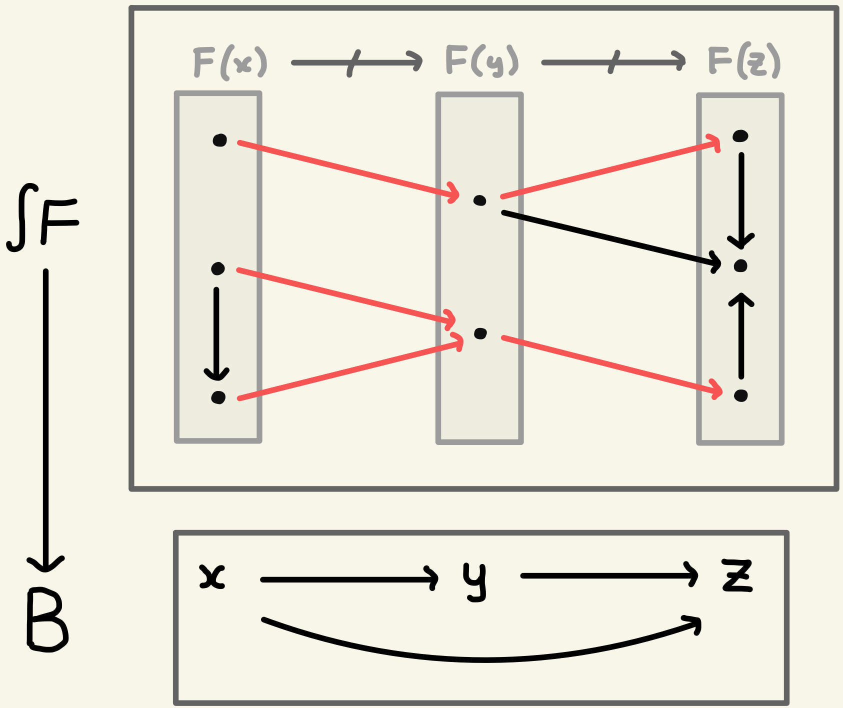
GLUING ALONG MONOS

Given a pair of monic lenses with common source, we can "glue" their target preordered sets together — this is called a pushout.



LENSES FROM MULTI-VALUED FUNCTIONS

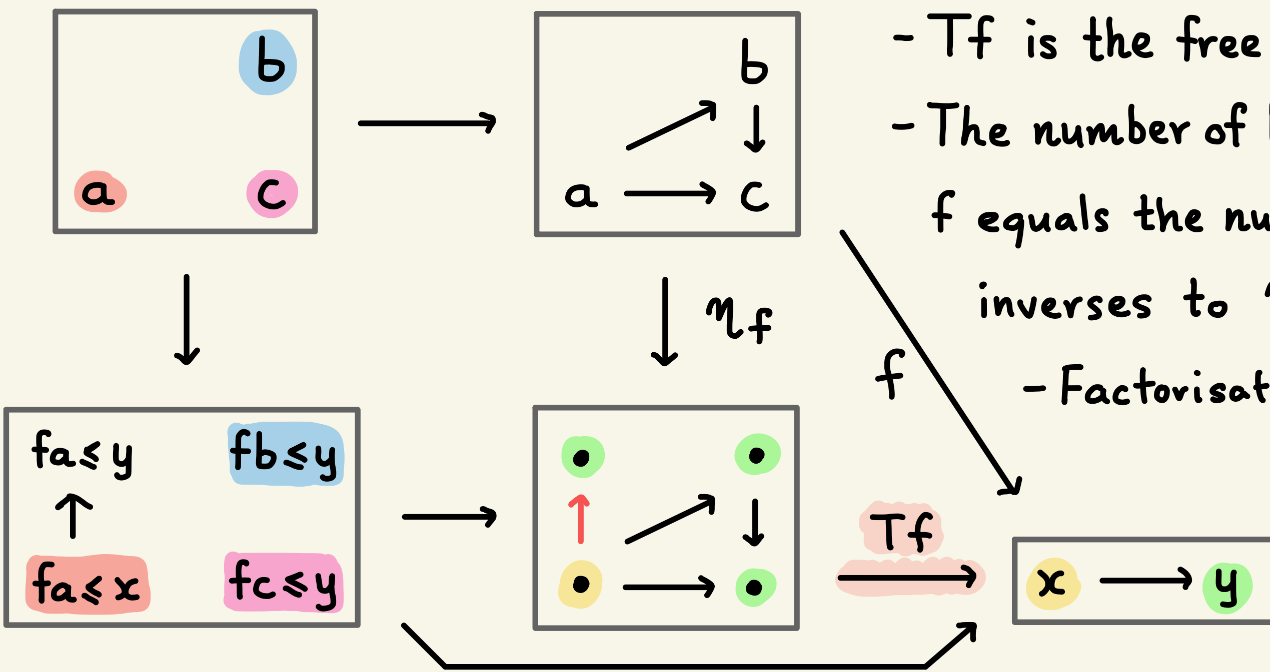
Given a preordered set B, how may we systematically construct a lens whose target is B? Using split multi-valued functions!



CONSTRUCTING FREE LENSES

Suppose we have an order-preserving function $f:(A,\leq)\longrightarrow(B,\leq)$; can we freely construct a lens using this data?

- Tf is the free lens induced by f.
- The number of lens structures on f equals the number of left inverses to η_f (+ conditions).
- Factorisation $f = Tf \circ \eta_f$.

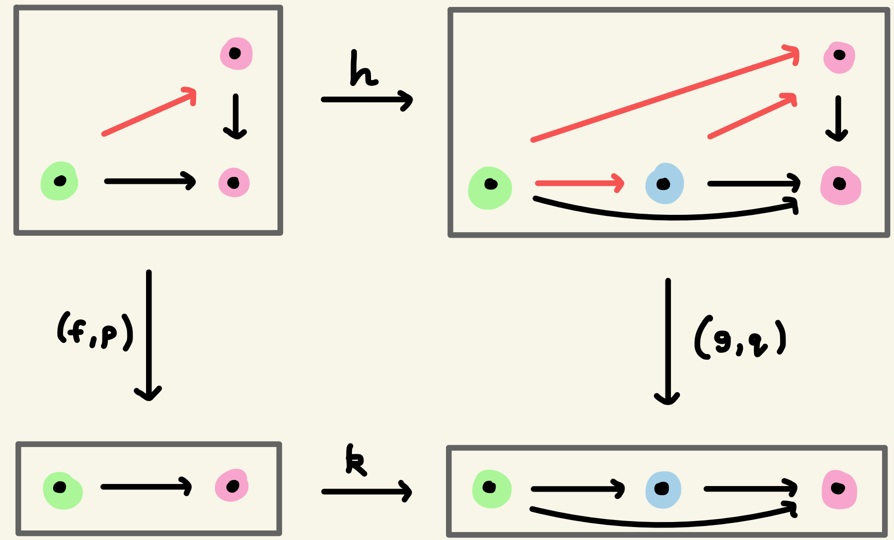


COMPARING LENSES

Suppose we have a pair of lenses, how might we compare them, or transform one into the other?

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,p)}"'] & B \arrow[d, "{(g,q)}"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

Take a pair of order-preserving functions such that $k \circ f = g \circ h$ and $h\,p(a,b) = q(ha, kb)$.



Here (h,k) includes a simple system into a more complex one while preserving the lens structure.

ENRICHMENT

A preordered set $(A, \leq)$ is a set A such that each pair $x, y \in A$ is assigned a truth value:

$$(x,y) \longmapsto \begin{cases} \top & \text{if } x \leq y \\ \bot & \text{otherwise} \end{cases}$$

What if we had a set of proofs instead?

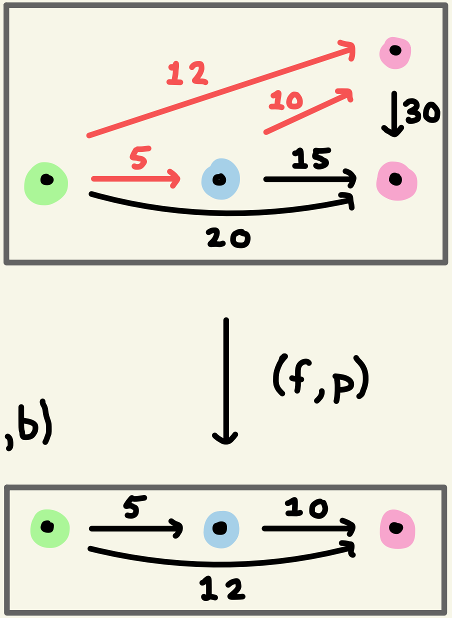
structure	enrichment
preorder	$\underline{A}(x,y)$ is a truth value
category	$\underline{A}(x,y)$ is a set
metric space	$\underline{A}(x,y)$ is a real number

Example: A system has a cost associated to each transistion between states.

Weighted lens

$$c(fx, fy) \leq c(x,y)$$

$$c(a, p(a,b)) = c(fa, b)$$



SUMMARY & FURTHER DIRECTIONS

- Lenses between preordered sets provide a versatile mathematical model for update synchronisation.
- Constructions involving lenses are developed with universal properties using tools from category theory.
- Framework for lenses naturally extends to cover more complex models via notion of enrichment.

- Can we compute all lenses between a fixed pair of preorders?
- How may these constructions be implemented in applications such as model-driven engineering?
- How can the study of lenses inform our understanding of other structures in category theory?