

THE RIGHT-CONNECTED COMPLETION

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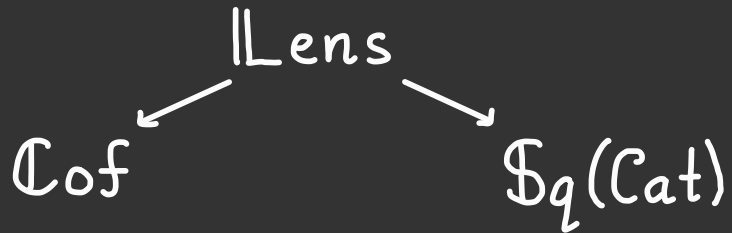
MOTIVATION: A.W.F.S.

- Algebraic weak factorisation systems (AWFS) are a generalisation of orthogonal factorisation systems.
- John Bourke and Richard Garner determined a characterisation of AWFS in terms of double categories with certain properties.
- This talk focuses on one of these properties: **right-connectedness**.
- **General motivation**: when can algebraic structure on a morphism be considered as a morphism in its own right?

MOTIVATION: LENSES

- **Lenses** are functors equipped with algebraic structure, and they are the **right class of an AWFS**.
- Many properties of lenses can be understood from studying the double category **LLens** whose:
 - objects are categories;
 - horizontal morphisms are functors;
 - vertical morphisms are lenses;
 - cells are "compatible squares".

- The algebraic structure on a functor which determines a lens is a **cofunctor**.
- The double category **LLens** is the **right-connected completion** of the double category **Cof**, and is the primary motivating example.



DEFINITION

- A **double category** is a (pseudo) category object in CAT.

$$\begin{array}{ccccc}
 & \xleftarrow{\text{dom}} & & \xleftarrow{\text{comp}} & \\
 \mathcal{D}_0 & \xrightarrow{\text{id}} & \mathcal{D}_1 & & \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \\
 & \xleftarrow{\text{cod}} & & &
 \end{array}$$

- A double category \mathcal{D} is called **right-connected** if its identity map is right adjoint to its codomain map.

$$\begin{array}{ccc}
 & \xleftarrow{\text{id}} & \\
 \mathcal{D}_0 & \xrightarrow{\top} & \mathcal{D}_1 \\
 & \xleftarrow{\text{cod}} &
 \end{array}$$

- Equivalently, if for every vertical morphism $f: A \rightarrow B$ there is a unique cell

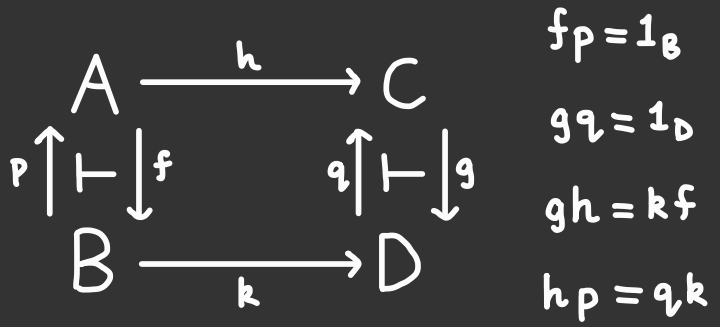
$$\begin{array}{ccc}
 A & \xrightarrow{\hat{f}} & B \\
 f \downarrow & \rho_f & \downarrow 1_B \\
 B & \xlongequal{\quad} & B
 \end{array}$$

such that for every cell α :

$$\begin{array}{ccc}
 A \xrightarrow{h} X & & A \xrightarrow{\hat{f}} B \xrightarrow{k} X \\
 f \downarrow \quad \alpha \quad \downarrow 1_X & = & f \downarrow \quad \rho_f \quad \downarrow 1_B \quad \downarrow 1_X \\
 B \xrightarrow{k} X & & B \xlongequal{\quad} B \xrightarrow{k} X
 \end{array}$$

EXAMPLES

- For any category \mathcal{C} , the double category of squares $\mathcal{S}q(\mathcal{C})$.
- For any 2-category \mathcal{K} , the double category $\mathcal{L}ali(\mathcal{K})$ of objects, 1-cells, and $lalis$ (left-adjoint left-inverse morphisms).



- The double category $\mathcal{Q}(\mathcal{K})$ of quintets is typically NOT right-connected: no universal property.
- The double category $\mathcal{S}Opf$ of categories, functors, split opfibrations, and commuting squares which preserve chosen lifts.
- If ID is right-connected, then its dual ID^{vh} is left-connected.

A USEFUL UNIT

- Let $RcDbl$ be the category of:
 - right-connected double cats
 - unitary double functors
- There is an adjunction

$$\begin{array}{ccc}
 & \xleftarrow{(-)_\circ} & \\
 \text{Cat} & \xrightleftharpoons[\mathbb{S}_q(-)]{\perp} & RcDbl
 \end{array}$$

with trivial counit and unit given by:

$$\begin{array}{ccc}
 ID & \xrightarrow{\eta_D} & \mathbb{S}_q(\mathcal{D}_\circ) \\
 \\
 \begin{array}{ccc}
 A & \xrightarrow{h} & B \\
 f \downarrow & \alpha & \downarrow g \\
 C & \xrightarrow{k} & D
 \end{array} & \longmapsto & \begin{array}{ccc}
 A & \xrightarrow{h} & B \\
 \hat{f} \downarrow & & \downarrow \hat{g} \\
 C & \xrightarrow{k} & D
 \end{array}
 \end{array}$$

- An **AWFS** is a right-connected double category such that

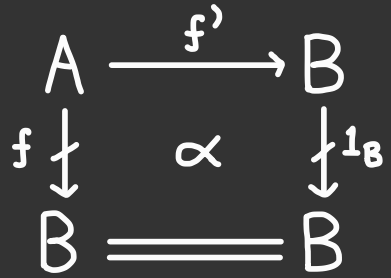
$$\mathcal{D}_1 \longrightarrow \mathbb{S}_q(\mathcal{D}_\circ)$$

is strictly monadic (\Rightarrow faithful).

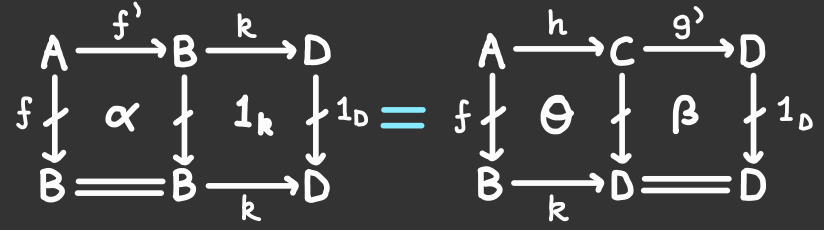
06 THE RIGHT-CONNECTED COMPLETION

The right-connected completion $\Gamma(\mathbb{D})$ of a double category \mathbb{D} has:

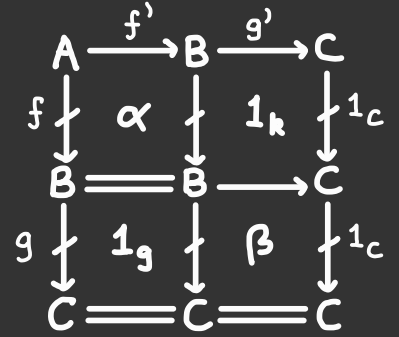
- objects & horizontal morphisms the same as \mathbb{D} ;
- vertical morphisms $(f, \alpha, f'): A \leftrightarrow B$ given by cells in \mathbb{D} :



- cells $\langle h, \theta, k \rangle: (f, \alpha, f') \rightarrow (g, \beta, g')$ given by cells θ in \mathbb{D} such that:



- vertical composition is given by:



THE UNIVERSAL PROPERTY

- Let DbL_{unit} be the category of:
 - double categories
 - unitary double functors
- There is an adjunction

$$\text{RcDbL} \begin{array}{c} \xleftarrow{\Gamma(-)} \\ \xrightarrow{\top} \\ \hookrightarrow \end{array} \text{DbL}_{\text{unit}}$$

with trivial unit and counit given by:

$$\Gamma(\text{ID}) \xrightarrow{\epsilon_{\text{ID}}} \text{ID}$$

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \downarrow (f, \alpha, f') & \Theta & \downarrow (g, \beta, g') \\ C & \xrightarrow{k} & D \end{array} \mapsto \begin{array}{ccc} A & \xrightarrow{h} & B \\ \downarrow f & \Theta & \downarrow g \\ C & \xrightarrow{k} & D \end{array}$$

The universal property states:

right-connected

$$\begin{array}{ccc} & & \Gamma(\text{ID}) \\ & \nearrow \exists! & \downarrow \epsilon_{\text{ID}} \\ \text{C} & \xrightarrow{F} & \text{ID} \end{array}$$

PROPERTIES

- If ID is a flat double category, then $\Gamma(ID)$ is flat.
- A horizontal morphism has a companion in $\Gamma(ID)$ if and only if it has a companion in ID .
- If ID is horizontally invariant, then so is $\Gamma(ID)$. Under this assumption, a horizontal morphism has a conjoint in $\Gamma(ID)$ if and only if it is invertible.
- If ID is a unit-pure (id is fully faithful) double category with tabulators (id has a right adjoint), then $\Gamma(ID)$ is unit-pure with tabulators.

EXAMPLES OF $\Gamma(\text{ID})$

• If ID is right-connected, then $\Gamma(\text{ID}) \cong \text{ID}$.

• If $\text{ID} = \mathcal{S}_q(\mathcal{C})^\vee$, then $\Gamma(\text{ID}) = \mathcal{S}\text{Epi}$,
the double cat of split epimorphisms:

$$\begin{array}{ccc}
 A & \xrightarrow{h} & C \\
 \varphi \uparrow \downarrow f & & \delta \uparrow \downarrow g \\
 B & \xrightarrow{k} & D
 \end{array}
 \quad
 \begin{array}{l}
 f \circ \varphi = 1_B \\
 g \circ \delta = 1_D
 \end{array}$$

• If $\text{ID} = \mathcal{C}\text{of}$, then $\Gamma(\text{ID}) = \mathcal{L}\text{ens}$, the
double category of categories, functors,
and delta lenses.

• If $\text{ID} = \mathcal{I}\text{pushout}(\mathcal{C})$, then
vertical morphisms in $\Gamma(\text{ID})$
are **epimorphisms**.

• If $\text{ID} = \mathcal{S}\text{pan}^{\text{vh}}$, then $\Gamma(\text{ID})^{\text{vh}}$
is $\mathcal{S}\text{Mult}$, the double cat of sets,
functions, and split multi-valued
functions.

• If $\text{ID} = \mathcal{Q}(\mathcal{K})$, then vertical
morphisms in $\Gamma(\text{ID})$ are **2-cells**.

LINK WITH COMPANIONS

- A double category ID is equipped with a functorial choice of companions if there is a strict horizontally trivial double functor:

$$(-)_* : \mathcal{S}_q(\mathcal{D}_0) \longrightarrow ID$$

- If ID is also right-connected, then there is an adjunction:

$$\begin{array}{ccc}
 & \xleftarrow{\eta_{ID}} & \\
 \mathcal{S}_q(\mathcal{D}_0) & \xrightleftharpoons[\perp]{} & ID \\
 & \xrightarrow{(-)_*} &
 \end{array}$$

- The right-connected completion is equipped with a functorial choice of companions if ID is:

$$\begin{array}{ccc}
 & \xrightarrow{(-)_*} & \Gamma(ID) \\
 \mathcal{S}_q(\mathcal{D}_0) & \xrightarrow{(-)_*} & ID \\
 & & \downarrow \epsilon_{ID}
 \end{array}$$

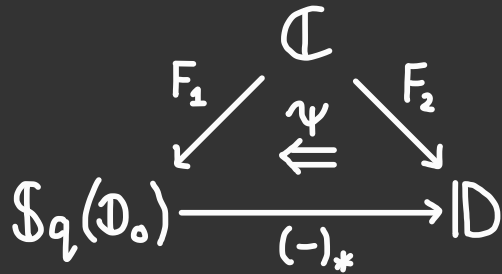
- Under these conditions we have:

$$\begin{array}{ccc}
 & \Gamma(ID) & \\
 \eta_{\Gamma(ID)} \swarrow & & \searrow \epsilon_{ID} \\
 \mathcal{S}_q(\mathcal{D}_0) & \xrightarrow{(-)_*} & ID \\
 & \xleftarrow{\varphi_{ID}} &
 \end{array}$$

φ_{ID} is globular

A BETTER UNIVERSAL PROPERTY

- Given a globular transformation between lax double functors

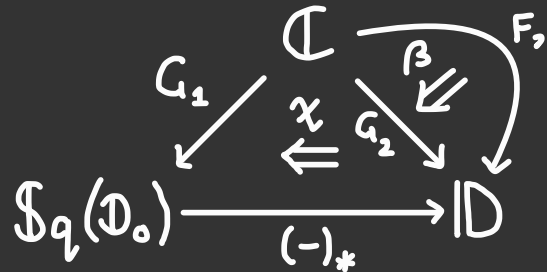
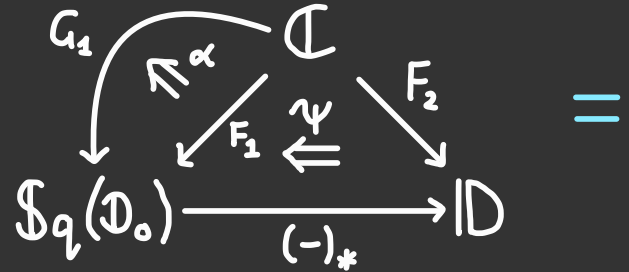


there exists a unique lax double functor $F: \mathcal{C} \rightarrow \mathbb{I}\Gamma(\text{ID})$ such that:

$$\varphi_{\text{ID}} F = \psi \quad \eta_{\mathbb{I}\Gamma(\text{ID})} F = F_1 \quad \varepsilon_{\text{ID}} F = F_2$$

$\mathbb{I}\Gamma(\text{ID})$ is the universal globular colax cone over $(-)_*$.

- Given transformations,

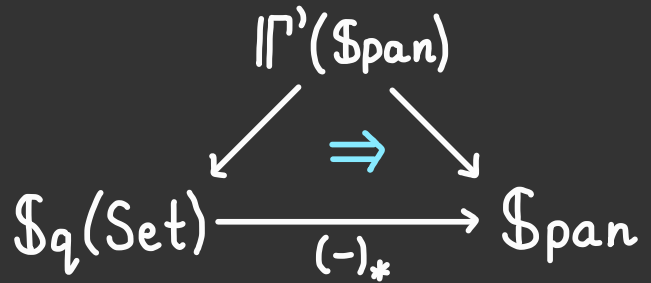


there is a unique $\delta: F \Rightarrow G$ s.t. :

$$\eta_{\mathbb{I}\Gamma(\text{ID})} \delta = \alpha \quad \varepsilon_{\text{ID}} \delta = \beta$$

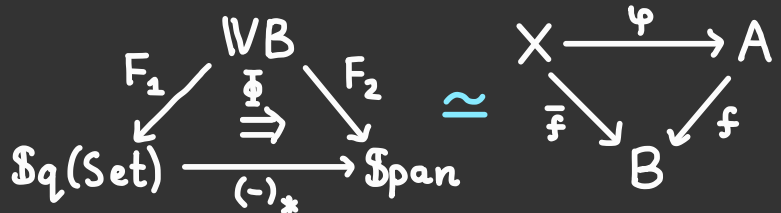
APPLICATION: GROTHENDIECK FOR LENSES

- Let $\mathbb{I}\Gamma'(ID) = \mathbb{I}\Gamma'(ID^{vh})^{vh}$ be the left-connected completion of ID .
- For $ID = \mathbb{S}pan$, we have the universal globular lax cone:



- Let $\mathbb{S}Mult = \mathbb{I}\Gamma'(\mathbb{S}pan)$ denote the double cat. of split multi-valued functions.

- There is a correspondance,



where Φ is globular, \bar{f} is a discrete opfibration and φ is id.on.obj.

- The Grothendieck construction for lenses is the right-to-left direction of the equivalence:
 $\mathbb{L}ens_{\mathbb{B}} \simeq [\mathbb{I}\Gamma'(\mathbb{S}pan), \mathbb{S}Mult]_{lax}$

A PAIR OF FORGETFUL FUNCTORS

$$\boxed{\text{ID}} \xleftarrow{\varepsilon_{\text{ID}}} \boxed{\Gamma(\text{ID})} \xrightarrow{\eta_{\Gamma(\text{ID})}} \boxed{\text{Sq}(\mathcal{D}_0)}$$

$$\begin{array}{ccccc}
 \mathcal{D}_1 & \xleftarrow{\mathfrak{L}} & \Gamma(\text{ID})_1 & \xrightarrow{\mathfrak{U}} & \text{Sq}(\mathcal{D}_0) \\
 \text{dom} \downarrow & \uparrow \text{id} & \downarrow \text{cod} & \uparrow \text{id} & \downarrow \text{cod} \\
 & | & & | & \\
 \mathcal{D}_0 & \xlongequal{\quad} & \mathcal{D}_0 & \xlongequal{\quad} & \mathcal{D}_0
 \end{array}$$

- Under what conditions does \mathfrak{L} have a right adjoint or does \mathfrak{U} have a left adjoint? When are they (co)monadic?

CONDITIONS FOR A RIGHT ADJOINT

Given a double category ID , suppose that:

(*) For each object $A \in D_0$, the fibre $\text{cod}^{-1}\{A\}$ of the functor $\text{cod}: D_1 \rightarrow D_0$ admits products with $1_A: A \rightarrow A$.

Theorem: If (*) holds, then $\mathcal{L}: \Gamma(ID)_1 \rightarrow D_1$ has a right adjoint.

Proof idea: Construct a functor $R: D_1 \rightarrow \Gamma(ID)_1$ as follows:

$$\begin{array}{c}
 A \\
 \downarrow f \\
 B
 \end{array}
 \mapsto
 \begin{array}{c}
 A \xleftarrow{\pi_A} P_f \xrightarrow{\pi_B} B \\
 \downarrow f \quad \text{counit} \downarrow \quad Rf \downarrow \quad 1_B \\
 B \xlongequal{\quad} B \xlongequal{\quad} B
 \end{array}
 \qquad
 \begin{array}{c}
 A \xlongequal{\quad} A \xrightarrow{f'} B \\
 \downarrow f \quad \quad \downarrow f \quad \alpha \downarrow \quad 1_B \\
 B \xlongequal{\quad} B \xlongequal{\quad} B
 \end{array}$$

1. Take the product.

2. Apply universal property for unit.

CONDITIONS FOR A LEFT ADJOINT

Given a double category ID , suppose:

(1) The domain map has a **lari**:

$$\begin{array}{ccc}
 & \xleftarrow{\text{dom}} & \\
 \mathcal{D}_0 & \xrightarrow{\tau} & \mathcal{D}_1 \\
 & \xrightarrow{\zeta} &
 \end{array}$$

(2) The codomain map is an **opfibration**.

Theorem. If (1) and (2) hold, then $\mathcal{U}: \Gamma(ID)_1 \rightarrow Sq(\mathcal{D}_0)$ has a left adjoint.

Proof idea: Construct a functor $F: Sq(\mathcal{D}_0) \rightarrow \Gamma(ID)_1$ as follows:

1. Take the transpose:

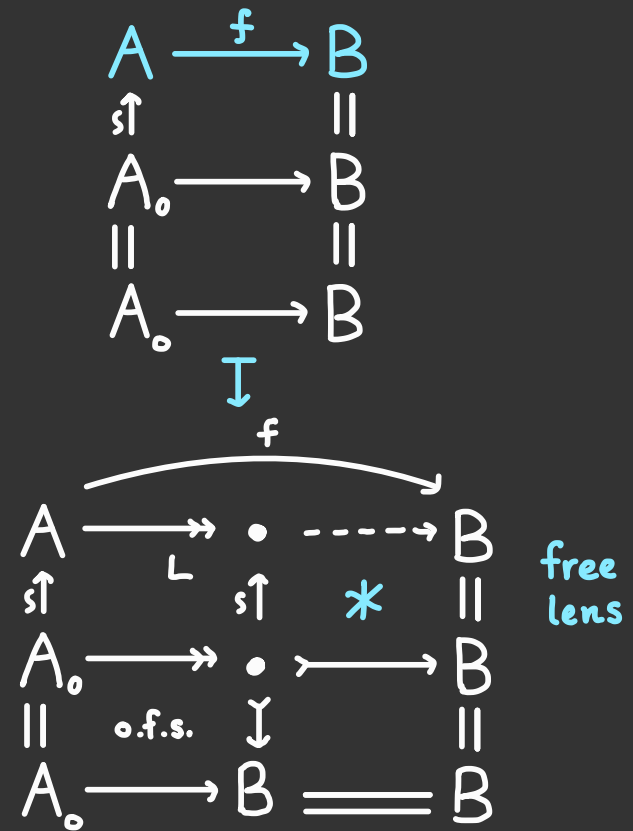
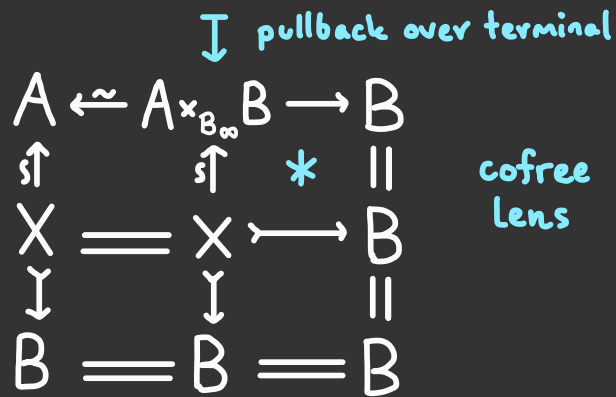
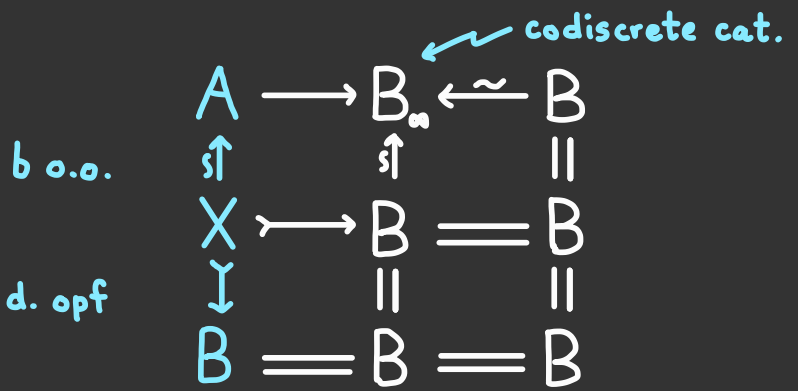
$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow & (f, 1_B)^t & \downarrow 1_B \\
 \zeta A & \longrightarrow & B
 \end{array}$$

2. Compute the opcartesian lift:

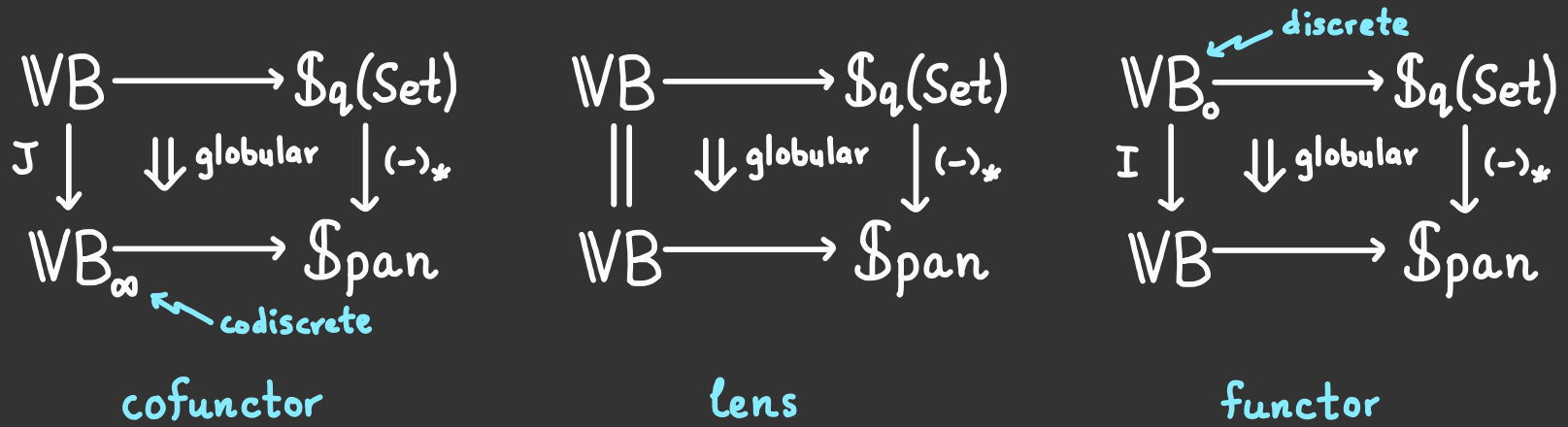
$$\begin{array}{ccccc}
 & & \xrightarrow{f} & & \\
 A & \xrightarrow{\eta_f} & \bullet & \xrightarrow{\mathcal{U}Ff} & B \\
 \downarrow & \text{lift} & \downarrow & Ff & \downarrow 1_B \\
 \zeta A & \longrightarrow & B & \equiv & B
 \end{array}$$

FREE & COFREE GENERALISED LENSES

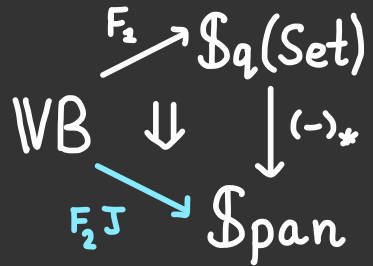
Let $\mathcal{Cof} \simeq \mathcal{S}pan(\mathcal{C}at, \mathcal{W}, \mathcal{M})$ for $\mathcal{W} = \{\text{b.o.o. functors}\}$ and $\mathcal{M} = \{\text{d. opfibrations}\}$.



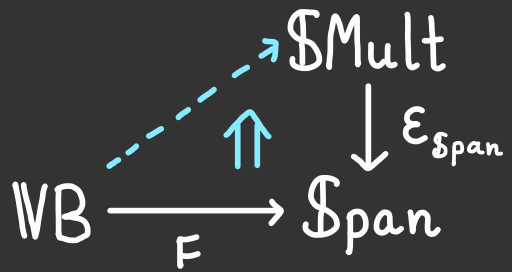
FREE & COFREE LENSES (AGAIN)



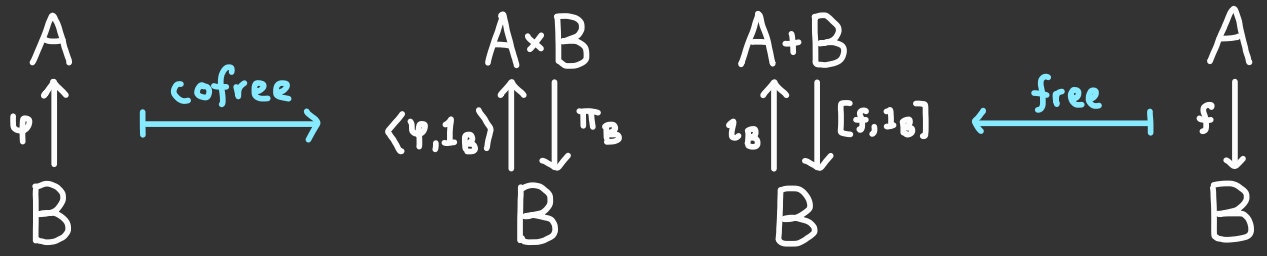
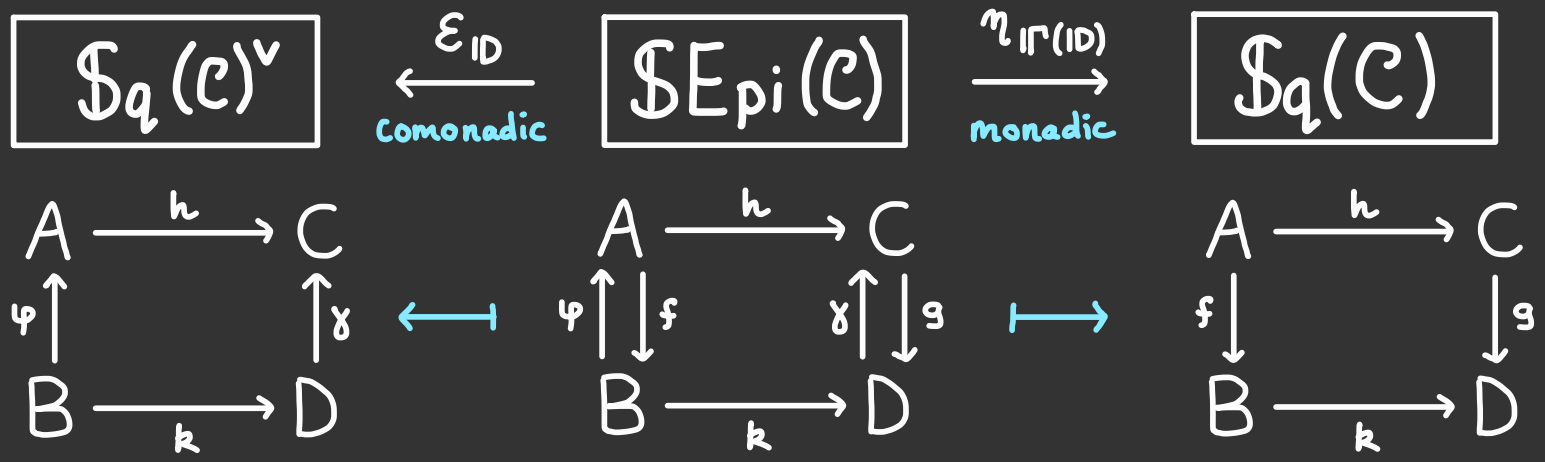
Cofree lens by composition:



Free lens by left Kan lift:



FREE & COFREE SPLIT EPIMORPHISMS



(CO) MONADICITY

$$\Gamma(\text{ID})_1 \xrightarrow{\mathcal{L}} \mathbb{D}_1$$

- Comonadicity is easy!
- If ID satisfies condition (*), which involves products, then \mathcal{L} has a right adjoint and is comonadic.

$$\begin{array}{ccccc}
 A & \xlongequal{\quad} & A & \xrightarrow{f'} & B \\
 f \downarrow & 1_f & f \downarrow & \alpha & \downarrow 1_B \\
 B & \xlongequal{\quad} & B & \xlongequal{\quad} & B
 \end{array}$$

Take span and apply universal property of product.

$$\Gamma(\text{ID})_1 \xrightarrow{\mathcal{U}} \text{Sq}(\mathbb{D}_0)$$

- Monadicity is hard.
- Can be proven ad hoc for particular examples such as $\text{ID} = \text{Cof}$ and $\text{ID} = \text{Sq}(\mathcal{C})^\vee$.

Open question: What are necessary or sufficient conditions for \mathcal{U} to be monadic?

SUMMARY & FUTURE WORK

- Introduced the **right-connected completion** of a double category.
- For ID equipped with functorial choice of companions, $\Pi(ID)$ has 2-dimensional universal property.
- Investigated conditions for adjoints to forgetful functors.
- Applied to the theory of lenses.
- Determine more examples of AWFS arising from right-connected comp.
- Find conditions on ID for monadicity.
- Is every double cat \mathcal{C} associated to an AWFS a double subcategory of some $\Pi(ID)$ for $ID \neq \mathcal{C}$?
- Explain coincidence that vertical morphism in $\Pi(\mathcal{C}of)$ is equivalent to lax double functor $\mathbb{W}B \rightarrow \Pi'(\mathcal{S}pan)$.