THE RIGHT-CONNECTED COMPLETION

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MOTIVATION: A.W.F.S.

- Algebraic weak factorisation systems (AWFS) are a generalisation of orthogonal factorisation systems.
- John Bourke and Richard Garner determined a characterisation of AWFS in terms of double categories with certain properties.
- This talk focuses on one of these properties: right-connectedness.
- General motivation: when can algebraic structure on a morphism be considered as a morphism in its own right?

MOTIVATION: LENSES

- Lenses are functors equipped with algebraic structure, and they are the right class of an AWFS.

- Many properties of lenses can be understood from studying the double category $\mathbb{L}\mathrm{ens}$ whose:
  - objects are categories;
  - horizontal morphisms are functors;
  - vertical morphisms are lenses;
  - cells are "compatible squares".

- The algebraic structure on a functor which determines a lens is a cofunctor.

- The double category $\mathbb{L}\mathrm{ens}$ is the right-connected completion of the double category $\mathbb{C}\mathrm{of}$, and is the primary motivating example.

$$\begin{tikzcd} & \mathbb{L}\mathrm{ens} \arrow[dl] \arrow[dr] & \\ \mathbb{C}\mathrm{of} & & \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at}) \end{tikzcd}$$

DEFINITION

- A **double category** is a (pseudo) category object in CAT.

$$
\begin{tikzcd}
\mathcal{D}_0 \arrow[r, "\mathrm{id}"] & \mathcal{D}_1 \arrow[l, shift right=2ex, swap, "\mathrm{dom}"] \arrow[l, shift left=2ex, "\mathrm{cod}"] & \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \arrow[l, swap, "\mathrm{comp}"]
\end{tikzcd}
$$

- A double category $\mathbb{D}$ is called **right-connected** if its identity map is right adjoint to its codomain map.

$$
\begin{tikzcd}
\mathcal{D}_0 \arrow[r, hook, shift left=1.5ex, "\mathrm{id}"] \arrow[r, phantom, "\top"] & \mathcal{D}_1 \arrow[l, shift left=1.5ex, "\mathrm{cod}"]
\end{tikzcd}
$$

- Equivalently, if for every vertical morphism $f : A \nrightarrow B$ there is a unique cell

$$
\begin{tikzcd}
A \arrow[r, "\hat{f}"] \arrow[d, "f"', "\shortmid" marking] \arrow[dr, phantom, "\rho_f"] & B \arrow[d, "1_B", "\shortmid" marking] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

such that for every cell α :

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "1_X", "\shortmid" marking] \\
B \arrow[r, "k"'] & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "\hat{f}"] \arrow[d, "f"', "\shortmid" marking] \arrow[dr, phantom, "\rho_f"] & B \arrow[r, "k"] \arrow[d, "\shortmid" marking] \arrow[dr, phantom, "1_k"] & X \arrow[d, "1_X", "\shortmid" marking] \\
B \arrow[r, equal] & B \arrow[r, "k"'] & X
\end{tikzcd}
$$

EXAMPLES

- For any category $\mathcal{C}$, the double category of squares $\mathbb{S}\mathrm{q}(\mathcal{C})$.

- For any 2-category $\mathcal{K}$, the double category $\mathbb{L}\mathrm{ali}(\mathcal{K})$ of objects, 1-cells, and lalis (left-adjoint left-inverse morphisms).

$$
\begin{tikzcd}
A \arrow[rr, "h"] \arrow[d, shift left=2, "f"] & & C \arrow[d, shift left=2, "g"] \\
B \arrow[u, shift left=2, "p"] \arrow[rr, "k"'] & & D \arrow[u, shift left=2, "q"]
\end{tikzcd}
\qquad
\begin{aligned}
fp &= 1_B \\
gq &= 1_D \\
gh &= kf \\
hp &= qk
\end{aligned}
$$

- The double category $\mathbb{Q}(\mathcal{K})$ of quintets is typically NOT right-connected: no universal property.

- The double category $\mathbb{S}\mathbb{O}\mathrm{pf}$ of categories, functors, split opfibrations, and commuting squares which preserve chosen lifts.

- If $\mathbb{D}$ is right-connected, then its dual $\mathbb{D}^{vh}$ is left-connected.

A USEFUL UNIT

- Let RcDbl be the category of:
  - right-connected double cats
  - unitary double functors

- There is an adjunction

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, hook, shift right, "\mathbb{S}q(-)"'] \arrow[r, phantom, "\perp"] & \mathrm{RcDbl} \arrow[l, shift right, "(-)_0"'] \end{tikzcd}$$

with trivial counit and unit given by:

$$\mathbb{D} \xrightarrow{\eta_{\mathbb{D}}} \mathbb{S}q(\mathbb{D}_0)$$

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"', "|" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g", "|" marking] \\ C \arrow[r, "k"'] & D \end{tikzcd} \longmapsto \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\hat{f}"', "|" marking] & B \arrow[d, "\hat{g}", "|" marking] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

- An AWFS is a right-connected double category such that

$$\mathbb{D}_1 \longrightarrow Sq(\mathbb{D}_0)$$

is strictly monadic ($\Rightarrow$ faithful).

THE RIGHT-CONNECTED COMPLETION

The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ of a double category $\mathbb{D}$ has:

- objects & horizontal morphisms the same as $\mathbb{D}$;
- vertical morphisms $(f, \alpha, f') : A \nrightarrow B$ given by cells in $\mathbb{D}$:

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "f"', "|" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "1_B", "|" marking] \\ B \arrow[r, equal] & B \end{tikzcd}$$

- cells $\langle h, \theta, k \rangle : (f, \alpha, f') \longrightarrow (g, \beta, g')$ given by cells θ in $\mathbb{D}$ such that:

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "f"', "|" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "k"] \arrow[d, "|" marking] \arrow[dr, phantom, "1_k"] & D \arrow[d, "1_D", "|" marking] \\ B \arrow[r, equal] & B \arrow[r, "k"'] & D \end{tikzcd} = \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"', "|" marking] \arrow[dr, phantom, "\theta"] & C \arrow[r, "g'"] \arrow[d, "|" marking] \arrow[dr, phantom, "\beta"] & D \arrow[d, "1_D", "|" marking] \\ B \arrow[r, "k"'] & D \arrow[r, equal] & D \end{tikzcd}$$

- vertical composition is given by:

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "f"', "|" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "g'"] \arrow[d, "|" marking] \arrow[dr, phantom, "1_k"] & C \arrow[d, "1_C", "|" marking] \\ B \arrow[r, equal] \arrow[d, "g"', "|" marking] \arrow[dr, phantom, "1_g"] & B \arrow[r] \arrow[d, "|" marking] \arrow[dr, phantom, "\beta"] & C \arrow[d, "1_C", "|" marking] \\ C \arrow[r, equal] & C \arrow[r, equal] & C \end{tikzcd}$$

THE UNIVERSAL PROPERTY

- Let $\mathsf{Dbl}_{\mathsf{unit}}$ be the category of:
  - double categories
  - unitary double functors
- There is an adjunction

$$
\begin{tikzcd}
\mathsf{RcDbl} \arrow[r, hook, shift right=2] \arrow[r, phantom, "\top"] & \mathsf{Dbl}_{\mathsf{unit}} \arrow[l, shift right=2, "\mathbb{\Gamma}(-)"']
\end{tikzcd}
$$

with trivial unit and counit given by:

$$\mathbb{\Gamma}(\mathbb{D}) \xrightarrow{\varepsilon_{\mathbb{D}}} \mathbb{D}$$

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "{(f,\alpha,f')}"', "\shortmid" marking] \arrow[dr, phantom, "\theta"] & B \arrow[d, "{(g,\beta,g')}", "\shortmid" marking] \\
C \arrow[r, "k"'] & D
\end{tikzcd}
\mapsto
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"', "\shortmid" marking] \arrow[dr, phantom, "\theta"] & B \arrow[d, "g", "\shortmid" marking] \\
C \arrow[r, "k"'] & D
\end{tikzcd}
$$

The universal property states:

right-connected

$$
\begin{tikzcd}
& \mathbb{\Gamma}(\mathbb{D}) \arrow[d, "\varepsilon_{\mathbb{D}}"] \\
\mathbb{C} \arrow[ur, dashed, "\exists!"] \arrow[r, "F"'] & \mathbb{D}
\end{tikzcd}
$$

PROPERTIES

- If $\mathbb{D}$ is a flat double category, then $\mathbb{\Gamma}(\mathbb{D})$ is flat.

- A horizontal morphism has a companion in $\mathbb{\Gamma}(\mathbb{D})$ if and only if it has a companion in $\mathbb{D}$.

- If $\mathbb{D}$ is horizontally invariant, then so is $\mathbb{\Gamma}(\mathbb{D})$. Under this assumption, a horizontal morphism has a conjoint in $\mathbb{\Gamma}(\mathbb{D})$ if and only if it is invertible.

- If $\mathbb{D}$ is a unit-pure (id is fully faithful) double category with tabulators (id has a right adjoint), then $\mathbb{\Gamma}(\mathbb{D})$ is unit-pure with tabulators.

EXAMPLES OF $\mathbb{\Gamma}(\mathbb{D})$

- If $\mathbb{D}$ is right-connected, then $\mathbb{\Gamma}(\mathbb{D}) \cong \mathbb{D}$.

- If $\mathbb{D} = \mathbb{S}q(\mathcal{C})^{\vee}$, then $\mathbb{\Gamma}(\mathbb{D}) = \mathbb{S}\mathrm{Epi}$, the double cat of split epimorphisms:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, shift left, "f"] & C \arrow[d, shift left, "g"] \\ B \arrow[r, "k"'] \arrow[u, shift left, "\varphi"] & D \arrow[u, shift left, "\gamma"] \end{tikzcd} \qquad \begin{array}{l} f \circ \varphi = 1_B \\ g \circ \gamma = 1_D \end{array}$$

- If $\mathbb{D} = \mathbb{C}\mathrm{of}$, then $\mathbb{\Gamma}(\mathbb{D}) = \mathbb{L}\mathrm{ens}$, the double category of categories, functors, and delta lenses.

- If $\mathbb{D} = \mathbb{P}\mathrm{ushout}(\mathcal{C})$, then vertical morphisms in $\mathbb{\Gamma}(\mathbb{D})$ are epimorphisms.

- If $\mathbb{D} = \mathbb{S}\mathrm{pan}^{vh}$, then $\mathbb{\Gamma}(\mathbb{D})^{vh}$ is $\mathbb{S}\mathrm{Mult}$, the double cat of sets, functions, and split multi-valued functions.

- If $\mathbb{D} = \mathbb{Q}(\mathcal{K})$, then vertical morphisms in $\mathbb{\Gamma}(\mathbb{D})$ are 2-cells.

LINK WITH COMPANIONS

- A double category $\mathbb{D}$ is equipped with a functorial choice of companions if there is a strict horizontally trivial double functor:

$$(-)_* : \mathbb{S}\mathrm{q}(\mathcal{D}_0) \longrightarrow \mathbb{D}$$

- If $\mathbb{D}$ is also right-connected, then there is an adjunction:

$$\begin{tikzcd}
\mathbb{S}\mathrm{q}(\mathcal{D}_0) \arrow[r, hook, bend right, "(-)_*"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend right, "\eta_{\mathbb{D}}"']
\end{tikzcd}$$

- The right-connected completion is equipped with a functorial choice of companions if $\mathbb{D}$ is:

$$\begin{tikzcd}
 & \mathbb{\Gamma}(\mathbb{D}) \arrow[d, "\varepsilon_{\mathbb{D}}"] \\
\mathbb{S}\mathrm{q}(\mathcal{D}_0) \arrow[ur, dashed, "(-)_*"] \arrow[r, "(-)_*"'] & \mathbb{D}
\end{tikzcd}$$

- Under these conditions we have:

$$\begin{tikzcd}
 & \mathbb{\Gamma}(\mathbb{D}) \arrow[dl, "\eta_{\mathbb{\Gamma}(\mathbb{D})}"'] \arrow[dr, "\varepsilon_{\mathbb{D}}"] \arrow[d, phantom, "\overset{\varphi_{\mathbb{D}}}{\Leftarrow}"] & \\
\mathbb{S}\mathrm{q}(\mathcal{D}_0) \arrow[rr, "(-)_*"'] & {} & \mathbb{D}
\end{tikzcd}$$

$\varphi_{\mathbb{D}}$ is globular

A BETTER UNIVERSAL PROPERTY

- Given a globular transformation between lax double functors

$$\begin{tikzcd}
& \mathbb{C} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] \arrow[d, phantom, "\overset{\psi}{\Leftarrow}"] & \\
\mathbb{S}\mathrm{q}(\mathbb{D}_0) \arrow[rr, "(-)_*"'] & & \mathbb{D}
\end{tikzcd}$$

there exists a unique lax double functor $F : \mathbb{C} \longrightarrow \mathbb{\Gamma}(\mathbb{D})$ such that:

$\varphi_{\mathbb{D}} F = \Psi \qquad \eta_{\mathbb{\Gamma}(\mathbb{D})} F = F_1 \qquad \varepsilon_{\mathbb{D}} F = F_2$

> $\mathbb{\Gamma}(\mathbb{D})$ is the **universal globular colax cone** over $(-)_*$.

- Given transformations,

$$\begin{tikzcd}
& \mathbb{C} \arrow[dl, bend right=60, "G_1"'] \arrow[dl, "F_1"] \arrow[dr, "F_2"] \arrow[d, phantom, "\overset{\psi}{\Leftarrow}"] \arrow[l, phantom, "\alpha"] & \\
\mathbb{S}\mathrm{q}(\mathbb{D}_0) \arrow[rr, "(-)_*"'] & & \mathbb{D}
\end{tikzcd}
=
\begin{tikzcd}
& \mathbb{C} \arrow[dl, "G_1"'] \arrow[dr, "G_2"'] \arrow[dr, bend left=60, "F_2"] \arrow[d, phantom, "\overset{\chi}{\Leftarrow}"] \arrow[r, phantom, "\beta"] & \\
\mathbb{S}\mathrm{q}(\mathbb{D}_0) \arrow[rr, "(-)_*"'] & & \mathbb{D}
\end{tikzcd}$$

there is a unique $\delta : F \Rightarrow G$ s.t. :

$\eta_{\mathbb{\Gamma}(\mathbb{D})} \delta = \alpha \qquad \varepsilon_{\mathbb{D}} \delta = \beta$

APPLICATION: GROTHENDIECK FOR LENSES

- Let $\mathbb{\Gamma}'(\mathbb{D}) = \mathbb{\Gamma}(\mathbb{D}^{vh})^{vh}$ be the left-connected completion of $\mathbb{D}$.

- For $\mathbb{D} = \mathbb{S}\mathrm{pan}$, we have the universal globular lax cone:

$$\begin{tikzcd}
 & \mathbb{\Gamma}'(\mathbb{S}\mathrm{pan}) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Rightarrow"] & \\
\mathbb{S}\mathrm{q}(\mathrm{Set}) \arrow[rr, "(-)_*"'] & {} & \mathbb{S}\mathrm{pan}
\end{tikzcd}$$

- Let $\mathbb{S}\mathrm{Mult} = \mathbb{\Gamma}'(\mathbb{S}\mathrm{pan})$ denote the double cat. of split multi-valued functions.

- There is a correspodance,

$$\begin{tikzcd}
 & \mathbb{V}B \arrow[dl, "F_1"'] \arrow[dr, "F_2"] \arrow[d, phantom, "\Phi \atop \Rightarrow"] & \\
\mathbb{S}\mathrm{q}(\mathrm{Set}) \arrow[rr, "(-)_*"'] & {} & \mathbb{S}\mathrm{pan}
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "\bar{f}"'] & & A \arrow[dl, "f"] \\
 & B &
\end{tikzcd}$$

where Φ is globular, $\bar{f}$ is a discrete opfibration and φ is id.on.obj.

- The Grothendieck construction for lenses is the right-to-left direction of the equivalence:

$$\mathcal{L}\mathrm{ens}_B \simeq [\mathbb{V}B, \mathbb{S}\mathrm{Mult}]_{\mathrm{lax}}$$

A PAIR OF FORGETFUL FUNCTORS

$$\boxed{\mathbb{D}} \xleftarrow{\varepsilon_{\mathbb{D}}} \boxed{\mathbb{\Gamma}(\mathbb{D})} \xrightarrow{\eta_{\mathbb{\Gamma}(\mathbb{D})}} \boxed{\mathbb{S}\mathrm{q}(\mathcal{D}_0)}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{D}_1 \arrow[d, shift right=6, "\mathrm{dom}"'] \arrow[d, shift left=6, "\mathrm{cod}"] & \Gamma(\mathbb{D})_1 \arrow[l, "\mathcal{L}"'] \arrow[r, "\mathcal{U}"] \arrow[d, shift right=6, "\mathrm{dom}"'] \arrow[d, shift left=6, "\mathrm{cod}"] & \mathrm{Sq}(\mathcal{D}_0) \arrow[d, shift right=6, "\mathrm{dom}"'] \arrow[d, shift left=6, "\mathrm{cod}"] \\
\mathcal{D}_0 \arrow[u, "\mathrm{id}"] \arrow[r, equal] & \mathcal{D}_0 \arrow[u, "\mathrm{id} \,\dashv"] \arrow[r, equal] & \mathcal{D}_0 \arrow[u, "\mathrm{id} \,\dashv"]
\end{tikzcd}$$

- Under what conditions does $\mathcal{L}$ have a right adjoint or does $\mathcal{U}$ have a left adjoint? When are they (co)monadic?

CONDITIONS FOR A RIGHT ADJOINT

Given a double category $\mathbb{D}$, suppose that:

(*) For each object $A \in D_0$, the fibre $\mathrm{cod}^{-1}\{A\}$ of the functor $\mathrm{cod}: D_1 \longrightarrow D_0$ admits products with $1_A : A \nrightarrow A$.

Theorem: If (*) holds, then $\mathcal{L} : \Gamma(\mathbb{D})_1 \longrightarrow \mathcal{D}_1$ has a right adjoint.

Proof idea: Construct a functor $R : \mathcal{D}_1 \longrightarrow \Gamma(\mathbb{D})_1$ as follows:

$$
\begin{tikzcd}
A \arrow[d, "f"', "|" marking] \\
B
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
A \arrow[d, "f"', "|" marking] \arrow[r, phantom, "\text{counit}"] & P_f \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] \arrow[d, "|" marking] \arrow[r, phantom, shift right=6ex, "Rf"] & B \arrow[d, "1_B", "|" marking] \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"', "|" marking] & A \arrow[r, "f'"] \arrow[d, "f"', "|" marking] \arrow[r, phantom, shift right=6ex, "\alpha"] & B \arrow[d, "1_B", "|" marking] \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}
$$

1. Take the product.

2. Apply universal property for unit.

CONDITIONS FOR A LEFT ADJOINT

Given a double category $\mathbb{D}$, suppose:

(1) The domain map has a lari:

$$
\begin{tikzcd}
\mathcal{D}_0 \arrow[r, hook, bend right, "Z"'] \arrow[r, phantom, "\top"] & \mathcal{D}_1 \arrow[l, bend right, "\mathrm{dom}"']
\end{tikzcd}
$$

(2) The codomain map is an opfibration.

Theorem. If (1) and (2) hold, then $\mathcal{U} : \Gamma(\mathbb{D})_1 \longrightarrow Sq(\mathcal{D}_0)$ has a left adjoint.

Proof idea: Construct a functor $\mathcal{F} : Sq(\mathcal{D}_0) \longrightarrow \Gamma(\mathbb{D})_1$ as follows:

1. Take the transpose:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "|" marking] \arrow[dr, phantom, "(f, 1_B)^t"] & B \arrow[d, "|" marking, "1_B"] \\
ZA \arrow[r] & B
\end{tikzcd}
$$

2. Compute the opcartesian lift:

$$
\begin{tikzcd}
A \arrow[r, "\eta_f"] \arrow[rr, bend left, "f"] \arrow[d, "|" marking] \arrow[dr, phantom, "\mathrm{lift}"] & \bullet \arrow[r, dashed, "\mathcal{U}\mathcal{F}f"] \arrow[d, "|" marking] \arrow[dr, phantom, "\mathcal{F}f"] \arrow[u, phantom, "\mathrm{unit}"] & B \arrow[d, "|" marking, "1_B"] \\
ZA \arrow[r] & B \arrow[r, equal] & B
\end{tikzcd}
$$

FREE & COFREE GENERALISED LENSES

Let $\mathbb{C}\mathrm{of} \simeq \mathbb{S}\mathrm{pan}(\mathcal{C}at, \mathcal{W}, \mathcal{M})$ for $\mathcal{W} = \{\text{b.o.o. functors}\}$ and $\mathcal{M} = \{\text{d opfibrations}\}$.

codiscrete cat.

$$
\begin{tikzcd}
A \arrow[r] & B_\infty & B \arrow[l, "\sim"'] \\
X \arrow[u, "s", "\text{b o.o.}"'] \arrow[r, tail] \arrow[d, "\text{d. opf}"'] & B \arrow[u, "s"] \arrow[r, equal] \arrow[d, equal] & B \arrow[u, equal] \arrow[d, equal] \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}
$$

$\mapsto$ pullback over terminal

$$
\begin{tikzcd}
A & A \times_{B_\infty} B \arrow[l, "\sim"'] \arrow[r] & B \\
X \arrow[u, "s"] \arrow[r, equal] \arrow[d] & X \arrow[u, "s"] \arrow[r, tail] \arrow[d] \arrow[ur, phantom, "*"] & B \arrow[u, equal] \arrow[d, equal] \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}
$$

cofree lens

$$
\begin{tikzcd}
A \arrow[r, "f"] & B \\
A_0 \arrow[u, "s"] \arrow[r] \arrow[d, equal] & B \arrow[u, equal] \arrow[d, equal] \\
A_0 \arrow[r] & B
\end{tikzcd}
$$

$\mapsto$

$$
\begin{tikzcd}
A \arrow[r, two heads, "L"'] \arrow[rr, bend left, "f"] & \bullet \arrow[r, dashed] & B \\
A_0 \arrow[u, "s"] \arrow[r, two heads] \arrow[d, equal] \arrow[dr, phantom, "\text{o.f.s.}"] & \bullet \arrow[u, "s"] \arrow[r, tail] \arrow[d] \arrow[ur, phantom, "*"] & B \arrow[u, equal] \arrow[d, equal] \\
A_0 \arrow[r] & B \arrow[r, equal] & B
\end{tikzcd}
$$

free lens

FREE & COFREE LENSES (AGAIN)

$$
\begin{tikzcd}
\mathbb{VB} \arrow[r] \arrow[d, "J"'] & \mathbb{S}\mathrm{q}(\mathbf{Set}) \arrow[d, "(-)_*"] \\
\mathbb{VB}_\infty \arrow[r] \arrow[ur, phantom, "\Downarrow \text{globular}"] & \mathbb{S}\mathrm{pan}
\end{tikzcd}
$$

codiscrete

cofunctor

$$
\begin{tikzcd}
\mathbb{VB} \arrow[r] \arrow[d, equal] & \mathbb{S}\mathrm{q}(\mathbf{Set}) \arrow[d, "(-)_*"] \\
\mathbb{VB} \arrow[r] \arrow[ur, phantom, "\Downarrow \text{globular}"] & \mathbb{S}\mathrm{pan}
\end{tikzcd}
$$

lens

discrete

$$
\begin{tikzcd}
\mathbb{VB}_0 \arrow[r] \arrow[d, "I"'] & \mathbb{S}\mathrm{q}(\mathbf{Set}) \arrow[d, "(-)_*"] \\
\mathbb{VB} \arrow[r] \arrow[ur, phantom, "\Downarrow \text{globular}"] & \mathbb{S}\mathrm{pan}
\end{tikzcd}
$$

functor

Cofree lens by composition:

$$
\begin{tikzcd}
 & \mathbb{S}\mathrm{q}(\mathbf{Set}) \arrow[d, "(-)_*"] \\
\mathbb{VB} \arrow[ur, "F_2"] \arrow[r, "F_2 J"'] \arrow[r, phantom, shift left=3, "\Downarrow"] & \mathbb{S}\mathrm{pan}
\end{tikzcd}
$$

Free lens by left Kan lift:

$$
\begin{tikzcd}
 & \mathbb{S}\mathrm{Mult} \arrow[d, "\varepsilon_{\mathrm{Span}}"] \\
\mathbb{VB} \arrow[ur, dashed] \arrow[r, "F"'] \arrow[r, phantom, shift left=3, "\Uparrow"] & \mathbb{S}\mathrm{pan}
\end{tikzcd}
$$

FREE & COFREE SPLIT EPIMORPHISMS

$$\boxed{\mathbb{S}q(\mathcal{C})^{\vee}} \xleftarrow[\text{comonadic}]{\varepsilon_{\mathbb{ID}}} \boxed{\mathbb{S}Epi(\mathcal{C})} \xrightarrow[\text{monadic}]{\eta_{\mathbb{I}\Gamma(\mathbb{ID})}} \boxed{\mathbb{S}q(\mathcal{C})}$$

$$\begin{tikzcd}
A \arrow[r, "h"] & C & \longleftarrow\!\!\mid & A \arrow[r, "h"] \arrow[d, shift left, "f"] & C \arrow[d, shift left, "g"] & \mid\!\!\longrightarrow & A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "k"'] \arrow[u, "\varphi"] & D \arrow[u, "\gamma"'] & & B \arrow[r, "k"'] \arrow[u, shift left, "\varphi"] & D \arrow[u, shift left, "\gamma"] & & B \arrow[r, "k"'] & D
\end{tikzcd}$$

$$\begin{tikzcd}
A & \overset{\text{cofree}}{\longmapsto} & A \times B \arrow[d, shift left, "\pi_B"] & A + B \arrow[d, shift left, "{[f, 1_B]}"] & \overset{\text{free}}{\longleftarrow\!\!\mid} & A \arrow[d, "f"] \\
B \arrow[u, "\varphi"] & & B \arrow[u, shift left, "{\langle \varphi, 1_B \rangle}"] & B \arrow[u, shift left, "\iota_B"] & & B
\end{tikzcd}$$

(CO)MONADICITY

$$\Gamma(\mathbb{D})_1 \xrightarrow{\mathcal{L}} \mathcal{D}_1$$

- Comonadicity is easy!

- If $\mathbb{D}$ satisfies condition $(*)$, which involves products, then $\mathcal{L}$ has a right adjoint and is comonadic.

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "1_f"] & A \arrow[r, "f'"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "1_B"] \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}$$

Take span and apply universal property of product.

$$\Gamma(\mathbb{D})_1 \xrightarrow{\mathcal{U}} \mathrm{Sq}(\mathcal{D}_0)$$

- Monadicity is hard.

- Can be proven ad hoc for particular examples such as $\mathbb{D} = \mathbb{C}\mathrm{of}$ and $\mathbb{D} = \mathrm{Sq}(\mathcal{C})^{\vee}$.

Open question: What are necessary or sufficient conditions for $\mathcal{U}$ to be monadic?

SUMMARY & FUTURE WORK

- Introduced the right-connected completion of a double category.
- For $\mathbb{D}$ equipped with functorial choice of companions, $\mathbb{\Gamma}(\mathbb{D})$ has 2-dimensional universal property.
- Investigated conditions for adjoints to forgetful functors.
- Applied to the theory of lenses.

- Determine more examples of AWFS arising from right-connected comp.
- Find conditions on $\mathbb{D}$ for monadicity.
- Is every double cat $\mathbb{C}$ associated to to an AWFS a double subcategory of some $\mathbb{\Gamma}(\mathbb{D})$ for $\mathbb{D} \neq \mathbb{C}$?
- Explain coincidence that vertical morphism in $\mathbb{\Gamma}(\mathbb{C}\mathrm{of})$ is equivalent to lax double functor $\mathbb{VB} \longrightarrow \mathbb{\Gamma}'(\mathbb{S}\mathrm{pan})$.