An introduction to delta lenses

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OVERVIEW OF THE TALK

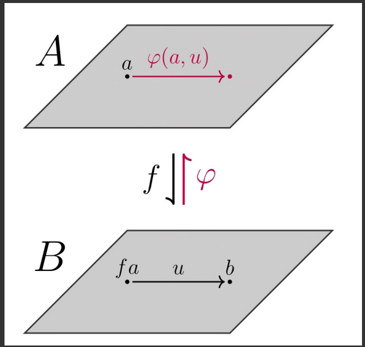
A delta lens is:

- A functor with additional structure
- A compatible functor & cofunctor
- A commutative triangle of functors

b.o.o. d.opf.

$$\begin{tikzcd} & \Lambda \arrow[dl, "\varphi"'] \arrow[dr, "\bar{f}"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

- A lax double functor into $\mathbb{S}$Mult
- A (co)algebra for a (co)monad
- Right class of an A.W.F.S.



$$\begin{tikzcd} & \boxed{\text{functor}} \arrow[dl, "\text{property}"'] \arrow[dr, "\text{structure}"] & \\ \boxed{\text{opfibration}} \arrow[dr, "\text{structure}"'] & & \boxed{\text{delta lens}} \arrow[dl, "\text{property}"] \\ & \boxed{\text{split opfibration}} & \end{tikzcd}$$

WHERE DO LENSES COME FROM?

Combinators for Bi-Directional Tree Transformations

A Linguistic Approach to the View Update Problem

J. Nathan Foster, Michael B. Greenwald, Jonathan T. Moore,
Benjamin C. Pierce, and Alan Schmitt

University of Pennsylvania

Proceedings POPL '05

3.1 Definition [Lenses]: A *lens* l comprises a partial function $l\nearrow$ from $\mathcal{U}$ to $\mathcal{U}$, called the *get function* of l, and a partial function $l\searrow$ from $\mathcal{U} \times \mathcal{U}$ to $\mathcal{U}$, called the *putback function.*

3.2 Definition [Well-behaved lenses]: Let l be a lens and let C and A be subsets of $\mathcal{U}$. We say that l is a *well behaved* lens from C to A, written $l \in C \rightleftharpoons A$, iff it maps arguments in C to results in A and vice versa

$$l\nearrow(C) \subseteq A \qquad \text{(GET)}$$
$$l\searrow(A \times C) \subseteq C \qquad \text{(PUT)}$$

and its *get* and *putback* functions obey the following laws:

$$l\searrow(l\nearrow c,\, c) \sqsubseteq c \qquad \text{for all } c \in C \qquad \text{(GETPUT)}$$
$$l\nearrow(l\searrow(a,\, c)) \sqsubseteq a \qquad \text{for all } (a, c) \in A \times C \qquad \text{(PUTGET)}$$

We call C the *source* and A the *target* in $C \rightleftharpoons A$.

A state-based lens $A \rightleftarrows B$ is a pair of functions,

$$f : A \longrightarrow B \qquad p : A \times B \longrightarrow A$$

satisfying the axioms:

1. $f\,p(a,b) = b$
2. $p(a, fa) = a$
3. $p(p(a,b), b') = p(a, b')$

$$\begin{array}{ccccc} A & & a & \longrightarrow & p(a,b) \\ \upharpoonleft\!\downharpoonright & & \vdots & & \vdots \\ B & & fa & \longrightarrow & b \end{array}$$

GENERALISING TO DELTA LENSES

From State- to Delta-Based Bidirectional Model Transformations: the Asymmetric Case

Zinovy Diskin[a] Yingfei Xiong[a] Krzysztof Czarnecki[a]

Definition 4 (delta lens)**.** A *delta lens* is a tuple $l = (\mathbf{A}, \mathbf{B}, \mathsf{get}, \mathsf{put})$, in which $\mathbf{A}$ and $\mathbf{B}$ are model spaces (i.e., connected categories) called the *source* and the *target* of the lens, $\mathsf{get}\colon \mathbf{A} \to \mathbf{B}$ is a graph morphism providing $\mathbf{B}$-views of $\mathbf{A}$-models and their deltas, and $\mathsf{put}\colon \mathbf{B}_1 \times \mathbf{A}_0 \to \mathbf{A}_1$ is a function translating view deltas back to the source so that laws PutInc_1 and PutInc_2 in Fig. 9 are respected.

A delta lens is called *well-behaved* (we will write wb) if it also satisfies GetId, PutId and PutGet laws. Particularly, GetId means that get is a semi-functor.

A wb delta lens is called *very well-behaved* if it satisfies GetGet and PutPut laws. Particularly, GetGet makes get a functor.

We will write $l\colon \mathbf{A} \rightleftarrows \mathbf{B}$ for a delta lens with source $\mathbf{A}$ and target $\mathbf{B}$, and denote the functions by get^l and put^l. We will often write *d-lens* for *delta lens*.

Equational laws		
(GetInc)	$\partial_{\mathsf{x}}(a.\mathsf{get}_1) = (\partial_{\mathsf{x}}a).\mathsf{get}_0,\ \mathsf{x} = \mathsf{s}, \mathsf{t}$	(1)
(PutInc$_1$)	$\mathsf{put}(b, A)$ is defined iff $A.\mathsf{get}_0 = \partial_{\mathsf{s}} b$	
(PutInc$_2$)	$\partial_{\mathsf{s}}\mathsf{put}(b, A) = A$	
(GetId)	$\mathsf{id}_A.\mathsf{get}_1 = \mathsf{id}_B$ with $B \stackrel{\text{def}}{=} A.\mathsf{get}_0$	(2)
(PutId)	$\mathsf{id}_A = \mathsf{put}(\mathsf{id}_B, A)$ with $B \stackrel{\text{def}}{=} A.\mathsf{get}_0$	
(PutGet)	$(\mathsf{put}(b, A)).\mathsf{get}_1 = b$	(3)
(GetGet)	$(a; a').\mathsf{get}_1 = (a.\mathsf{get}_1); (a'.\mathsf{get}_1)$	(4)
(PutPut)	$\mathsf{put}(b; b', A) = \mathsf{put}(b, A); \mathsf{put}(b', A')$ with $A' \stackrel{\text{def}}{=} \partial_{\mathsf{t}}\mathsf{put}(b, A)$	

A **delta lens** $(f,\varphi)\colon A \rightleftarrows B$ consists of a functor $f\colon A \longrightarrow B$ together with a lifting operation,

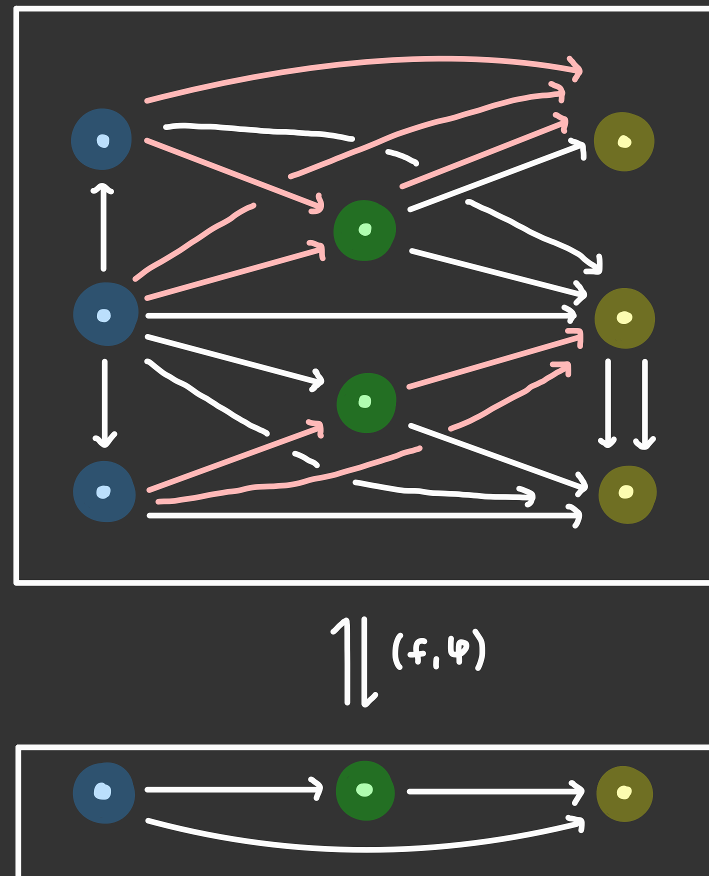
$$\begin{tikzcd} A \arrow[d, shift left, "(f{,}\varphi)"'] & a \arrow[r, "\varphi(a{,}u)"] \arrow[d, phantom, "\vdots"] & a' \arrow[d, phantom, "\vdots"] \\ B \arrow[u, shift left] & fa \arrow[r, "u"] & b \end{tikzcd}$$

satisfying the axioms:

1. $f\varphi(a,u) = u$
2. $\varphi(a, 1_{fa}) = 1_a$
3. $\varphi(a, v\circ u) = \varphi(a', v)\circ\varphi(a,u)$

EXAMPLES OF DELTA LENSES

- State-based lens $\simeq$ delta lens between codiscrete categories.
- Split opfibration $\simeq$ delta lens such that $\varphi(a,u)$ are opcartesian.
- Discrete opfibration $\simeq$ delta lens such that $\varphi(a,u)$ are unique.
- Spilt epimorphism of monoids with a chosen section $\simeq$ delta lens between monoids.



DELTA LENSES VIA COFUNCTORS

A **cofunctor** $(f, \varphi) : A \nrightarrow B$ consists of a function $f : \mathrm{Obj}(A) \longrightarrow \mathrm{Obj}(B)$ together with a lifting operation,

$$\begin{tikzcd} A \arrow[d, "{(f,\varphi)}"'] & a \arrow[r, "{\varphi(a,u)}"] \arrow[d, phantom, "\vdots"] & a' \arrow[d, phantom, "\vdots"] \\ B & fa \arrow[r, "u"] & b \end{tikzcd}$$

satisfying the axioms:

1. $f\,\mathrm{cod}(\varphi(a,u)) = \mathrm{cod}(u)$ *
2. $\varphi(a, 1_{fa}) = 1_a$
3. $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a, u)$

Ahman & Uustalu: A delta lens $(f, \varphi) : A \rightleftarrows B$ is equivalent to a **functor** $f : A \longrightarrow B$ and a **cofunctor** $(f_0, \varphi) : A \nrightarrow B$ such that $f \varphi(a,u) = u$.

Motto: A delta lens is a compatible functor and cofunctor pair.

A DOUBLE CATEGORY OF COFUNCTORS

A square of functors and cofunctors,

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,\varphi)}"'] & C \arrow[d, "{(g,\gamma)}"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

is compatible if:

(1) For all $a \in A$, $gha = kfa$

(2) For all $(a \in A, u : fa \to b \in B)$,

$$h\varphi(a,u) = \gamma(ha, ku)$$

Let $\mathcal{C}of$ denote the category of cofunctors and compatible squares.

Let $\mathbb{C}of$ be the double category whose:

- objects are categories
- horizontal morphisms are functors
- vertical morphisms are cofunctors
- cells are compatible squares

A delta lens is a cell in $\mathbb{C}of$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "{(f,\varphi)}"'] & B \arrow[d, "1_B"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

THE RIGHT-CONNECTED COMPLETION

The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ is a double category whose:

- objects and hori. morphisms are those of $\mathbb{D}$.
- vertical morphisms (f, α, f') are cells in $\mathbb{D}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f'"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "1_B"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

- cells θ are those of $\mathbb{D}$ such that:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\theta"] & C \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\beta"] & D \arrow[d, "1_D"] \\ B \arrow[r, "k"'] & D \arrow[r, equal] & D \end{tikzcd} = \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f'"'] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "1_k"] & D \arrow[d, "1_D"] \\ B \arrow[r, equal] & B \arrow[r, "k"'] & D \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{RCDbl} \arrow[r, hook, bend right] \arrow[r, phantom, "\top"] & \mathsf{Dbl}_{unit} \arrow[l, bend right, "\mathbb{\Gamma}(-)"'] \end{tikzcd}$$

Canonical double functors:

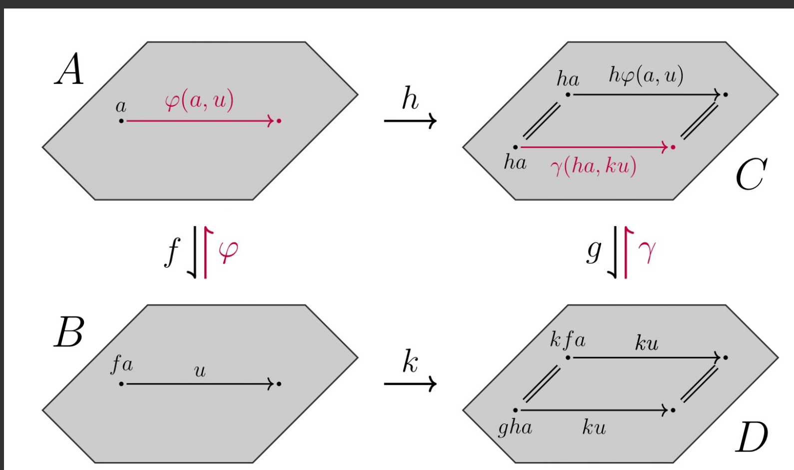
$$\mathbb{D} \longleftarrow \mathbb{\Gamma}(\mathbb{D}) \longrightarrow \mathbb{S}\mathrm{q}(D_0)$$

$$\begin{tikzcd} A \arrow[d, "f'"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "{(f,\alpha,f')}"] \\ B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$$

A DOUBLE CATEGORY OF DELTA LENSES

Let $\mathbb{L}\mathrm{ens}$ be the double cat. whose:

- objects are categories
- horizontal morphisms are functors
- vertical morphisms are delta lenses
- cells are given by:



Theorem: $\mathbb{L}\mathrm{ens} \simeq \mathbb{\Gamma}(\mathbb{C}\mathrm{of})$

A delta lens is a vertical morphism in the right-connected completion of $\mathbb{C}\mathrm{of}$.

Canonical double functors:

$$\mathbb{C}\mathrm{of} \longleftarrow \mathbb{L}\mathrm{ens} \longrightarrow \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at})$$

$$A \xrightarrow{(f_0, \varphi)} B \qquad A \overset{(f, \varphi)}{\rightleftarrows} B \qquad A \xrightarrow{f} B$$

TABULATORS & SPAN REPRESENTABILITY

A tabulator is a certain double-categorical limit.

$$\begin{tikzcd} Tf \arrow[r, "\pi_A"] \arrow[d, "1"', "\shortmid" marking] \arrow[dr, phantom, "\pi_f"] & A \arrow[d, "f", "\shortmid"' marking] \\ Tf \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

The double category $\mathbb{C}\mathrm{of}$ has:

- tabulators
- $\mathrm{Hor}_0(\mathbb{C}\mathrm{of}) = \mathcal{C}\mathrm{at}$ has pullbacks
- Fully faithful pseudo double fun.

$$\mathbb{C}\mathrm{of} \longrightarrow \mathbb{S}\mathrm{pan}(\mathcal{C}\mathrm{at})$$

A cofunctor $(f, \varphi) \colon A \nrightarrow B \simeq$ span of functors:

$$\begin{tikzcd} A & \Lambda \arrow[l, "\varphi"'] \arrow[r, "\bar{f}"] & B \\ a \arrow[d, "\varphi(a,u)"'] & a \arrow[d, "(a,u)"] & fa \arrow[d, "u"] \\ a' & a' & b \end{tikzcd}$$

A delta lens $(f, \varphi) \colon A \rightleftarrows B \simeq$

b.o.o.

d.opf.

$$\begin{tikzcd} & \Lambda \arrow[dl, "\varphi"'] \arrow[dr, "\bar{f}"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

SPLIT OPFIBRATIONS VIA DÉCALAGE

The décalage construction is a comonad (D, ε, δ) on Cat given by:

$$DA = \sum_{a \in A} A/a$$

Given a lens $(f, \varphi) \colon A \rightleftarrows B$ construct:

$$\begin{tikzcd}
\Lambda \times_A DA \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[rr, bend left, "Df \circ \pi_1"] & DA \arrow[r, "Df"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\text{nat.}"] & DB \arrow[d, "\varepsilon"] \\
\Lambda \arrow[r, "\varphi"'] \arrow[rr, bend right, "\overline{f}"'] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

A lens (f, φ) is a split opfibration iff $Df \circ \pi_1$ is a discrete opfibration.

$$\begin{tikzcd}
DA \arrow[d, "Df"'] \\
DB
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[rr, "\varphi(a,u)"] \arrow[dr, "\omega"'] & & a'' \arrow[dl, dashed, "\exists!"] \\
& a' & \\
fa \arrow[rr, "u"] \arrow[dr, "f\omega"'] & & b \arrow[dl, "v"] \\
& fa' &
\end{tikzcd}$$

A GROTHENDIECK CONSTRUCTION FOR LENSES

The classical Grothendieck construction is an equivalence:

$$\mathsf{SOpf}(B) \simeq [B, \mathcal{C}\mathsf{at}]$$

A special case is given by:

$$\mathsf{DOpf}(B) \simeq [B, \mathcal{S}\mathsf{et}]$$

For arbitrary functors we have:

$$\mathcal{C}\mathsf{at}/B \simeq [B, \mathbb{S}\mathsf{pan}]_{\mathsf{lax}}$$

Let $\mathcal{L}\mathsf{ens}(B)$ be the fibre over B of the codomain map $\mathrm{cod} \colon \mathcal{L}\mathsf{ens} \longrightarrow \mathcal{C}\mathsf{at}$ of the double category $\mathbb{L}\mathsf{ens}$.

Let $\mathbb{S}\mathbb{M}\mathsf{ult}$ be the double category of sets, functions, and split multi-valued functions

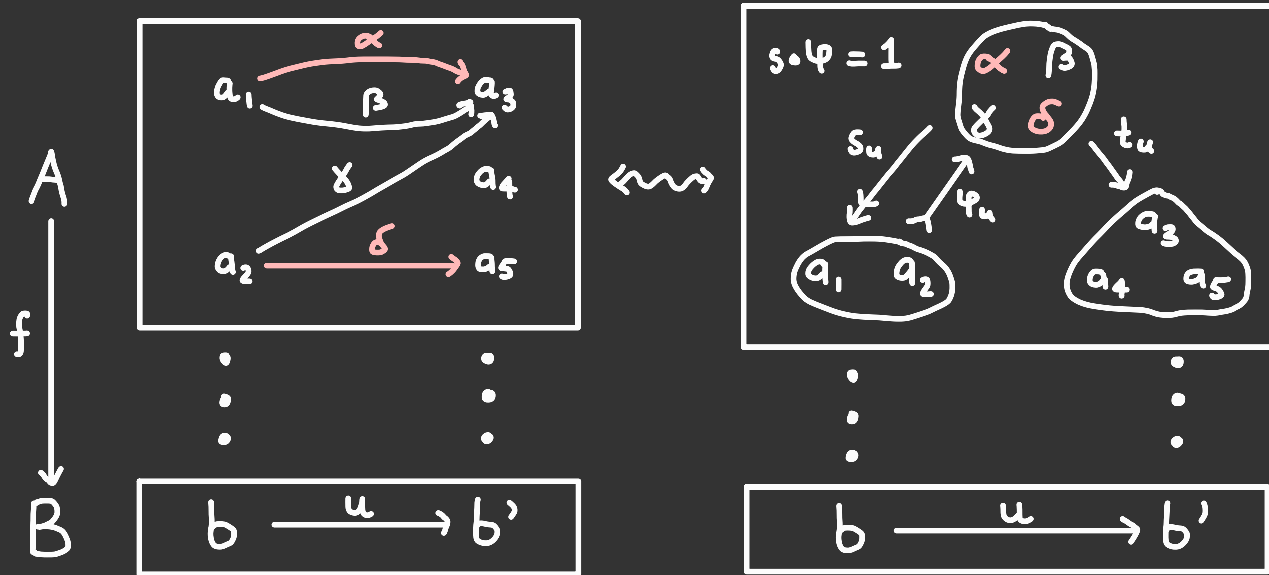
$$\begin{tikzcd} A \arrow[r, shift left, tail, "\varphi"] & X \arrow[l, shift left, two heads, "s"] \arrow[r, "t"] & B \end{tikzcd}$$

There are double functors:

$$\begin{tikzcd} \mathbb{S}\mathsf{pan} & \mathbb{S}\mathbb{M}\mathsf{ult} \arrow[l] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathbb{S}\mathsf{q}(\mathcal{S}\mathsf{et}) \arrow[l, shift right, hook'] \end{tikzcd}$$

Theorem: $\mathcal{L}\mathsf{ens}(B) \simeq [B, \mathbb{S}\mathbb{M}\mathsf{ult}]_{\mathsf{lax}}$

FIBRED APPROACH TO LENSES



For each $u : b \to b'$ in B, there is a split multi-valued function between the fibres. We have $\varphi_u(a_1) = \alpha$ and $\varphi_u(a_2) = \delta$.

A UNIVERSAL PROPERTY VIA COMPANIONS

If $\mathbb{D}$ is equipped with a functorial choice of companions, there is a strict double functor:

$$\mathbb{S}q(D_0) \xrightarrow{(-)_*} \mathbb{D}$$

The left-connected completion is a universal lax (globular) cone over this double functor:

$$\begin{tikzcd} & \mathbb{\Gamma}'(\mathbb{D}) \arrow[dl, "\mathcal{L}"'] \arrow[dr, "\mathcal{R}"] \arrow[d, phantom, "\Rightarrow"] & \\ \mathbb{S}q(D_0) \arrow[rr, "(-)_*"'] & {} & \mathbb{D} \end{tikzcd}$$

$$\frac{\text{lax double functors } \mathbb{B} \xrightarrow{F} \mathbb{\Gamma}'(\mathbb{D})}{\text{globular transformations} \quad \begin{tikzcd} & \mathbb{B} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] \arrow[d, phantom, "\Rightarrow"] & \\ \mathbb{S}q(D_0) \arrow[rr, "(-)_*"'] & {} & \mathbb{D} \end{tikzcd}}$$

If $\mathbb{D} = \mathbb{S}pan$, then $\mathbb{S}Mult = \mathbb{\Gamma}'(\mathbb{D})$ and lenses into $\mathbb{B}$ correspond to:

$$\begin{tikzcd} & \mathbb{B} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & \\ \mathbb{S}q(\mathsf{Set}) \arrow[rr, "(-)_*"'] & {} & \mathbb{S}pan \end{tikzcd}$$

MORE DOUBLE-CATEGORICAL PROPERTIES

- Codomain map $\mathrm{cod}\colon \mathcal{L}\mathrm{ens} \longrightarrow \mathcal{C}\mathrm{at}$ is a bifibration $\Rightarrow$ change of base

$$\begin{tikzcd} \mathcal{L}\mathrm{ens}(B) \arrow[r, shift right, "\Delta_g"'] \arrow[r, phantom, "\perp"] & \mathcal{L}\mathrm{ens}(C) \arrow[l, shift right, "\Sigma_g"'] \end{tikzcd}$$

- $\mathcal{L}\mathrm{ens} \longrightarrow \mathcal{C}\mathrm{of}$ is comonadic; "lenses are coalgebras for a comonad on cofunctors"
- $\mathcal{L}\mathrm{ens} \longrightarrow \mathrm{Sq}(\mathcal{C}\mathrm{at})$ is monadic; "lenses are algebras for a monad on functors"
- $\mathbb{L}\mathrm{ens} \simeq \mathbb{\Gamma}(\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan}))$ "lenses are monad morphisms"
  - $\mathbb{S}\mathrm{pan}(\mathcal{E})$ – internal lenses
  - $\mathbb{M}\mathrm{at}(\mathcal{V})$ – enriched lenses
- An algebraic weak factorisation system with $R\text{-}\mathbb{A}\mathrm{lg} \cong \mathbb{L}\mathrm{ens}$
- Companions in $\mathbb{L}\mathrm{ens}$ are d.opfs.
- $\mathbb{L}\mathrm{ens}$ is symmetric monoidal.
- Can complete lenses to s.opf.

$$\begin{tikzcd} \mathrm{SOpf}(B) \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathcal{L}\mathrm{ens}(B) \arrow[l, shift right] \end{tikzcd}$$