Constructing lenses in double categories

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MOTIVATION

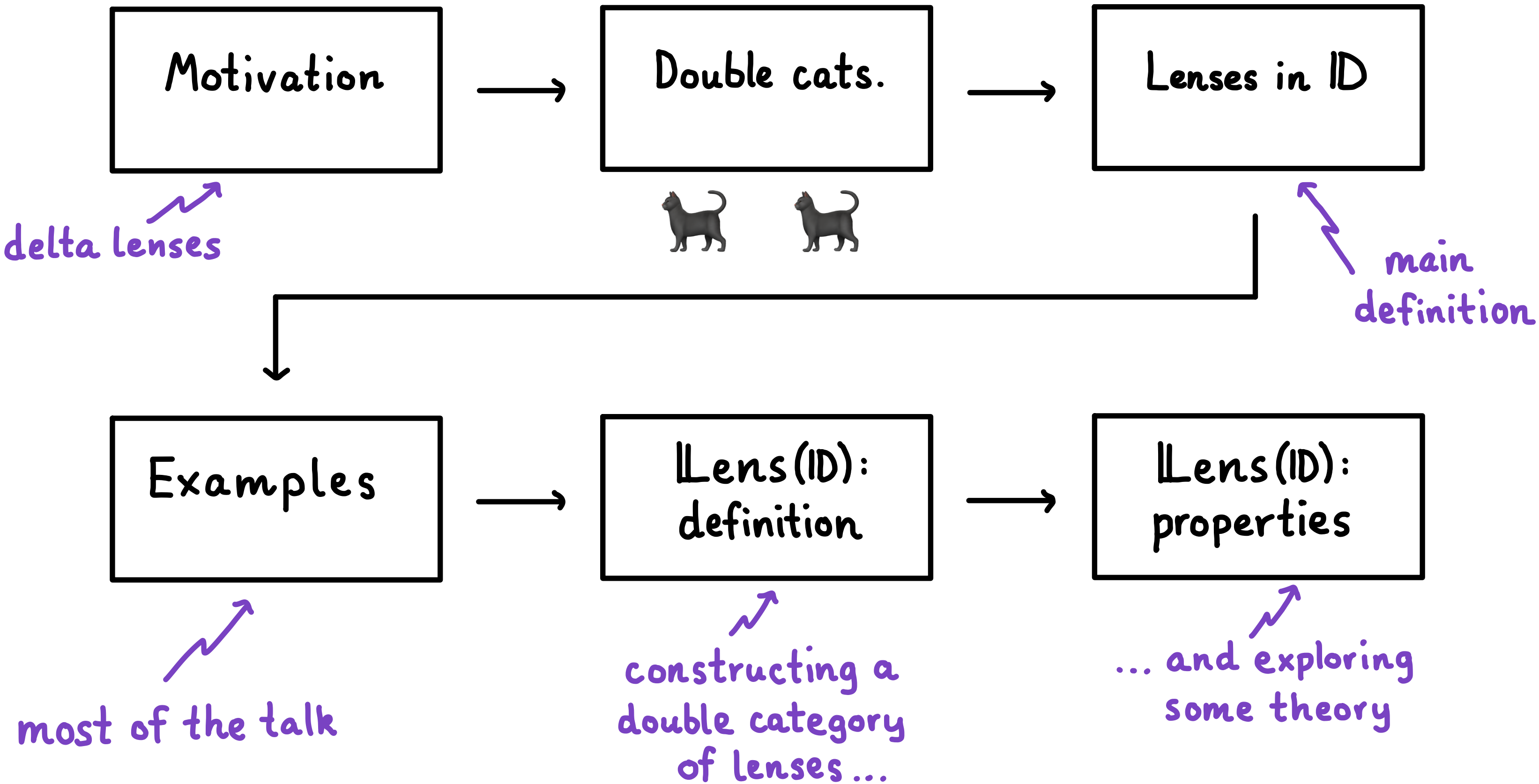
- A lens is a morphism consisting of two components:

$$\begin{tikzcd} \boxed{A} \arrow[r, shift right, "\text{forwards}"'] & \boxed{B} \arrow[l, shift right, "\text{backwards}"', "|" marking] \end{tikzcd}$$

- What if the forwards and backwards components of a lens were the horizontal and vertical morphisms in a double category?

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "|" marking] \arrow[dr, phantom, "\alpha"] & A \arrow[d, equal] \\ B \arrow[r] & A \end{tikzcd}$$

OVERVIEW OF THE TALK



MOTIVATING EXAMPLE: FUNCTORS & COFUNCTORS

A cofunctor $(f, \varphi) : A \nrightarrow B$ consists of an object assignment,

$$f : \mathrm{Obj}(A) \longrightarrow \mathrm{Obj}(B)$$

and a lifting operation,

$$\begin{tikzcd}
A \arrow[d, "{(f,\varphi)}"'] & a \arrow[r, "{\varphi(a,u)}"] \arrow[d, phantom, "\vdots"] & a' \arrow[d, phantom, "\vdots"] \\
B & fa \arrow[r, "u"] & b = fa'
\end{tikzcd}$$

where $a' = \mathrm{cod}(\varphi(a,u))$, such that:

(1) $\varphi(a, 1_{fa}) = 1_a$

(2) $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a, u)$

A delta lens $A \longrightarrow B$ consists of a functor $f : A \longrightarrow B$ and a cofunctor $(f', \varphi) : A \longrightarrow B$ such that:

(i) $f'a = fa$ for all $a \in A$

(ii) $f\varphi(a,u) = u$ for all $(a, u : fa \to b)$.

<u>Main idea</u>: A lens is a compatible pair of morphisms (PutGet law).

How can we express compatibility?

WHAT IS A DOUBLE CATEGORY?

A double category $\mathbb{D}$ consists of:

- objects A, B, C, D, etc
- horizontal morphisms $\bullet \longrightarrow \bullet$
- vertical morphisms $\bullet \xrightarrow{\shortmid} \bullet$
- cells

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"', "\shortmid"{marking}] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g", "\shortmid"{marking}] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

Horizontal composition is strict,

and vertical composition is weak.

Identity cells look like:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "1_A"', "\shortmid"{marking}] \arrow[dr, phantom, "1_h"] & C \arrow[d, "1_C", "\shortmid"{marking}] \\ A \arrow[r, "h"'] & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"', "\shortmid"{marking}] \arrow[dr, phantom, "1_f"] & A \arrow[d, "f", "\shortmid"{marking}] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Composition of cells looks like:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta"] & \bullet \arrow[rr] \arrow[d] \arrow[drr, phantom, "\gamma"] & & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\delta"] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[d] \arrow[drr, phantom, "\varepsilon"] & \bullet \arrow[r] & \bullet \arrow[rr] \arrow[d] \arrow[drrr, phantom, "\eta"] & & \bullet \arrow[r] & \bullet \arrow[d] \\ \bullet \arrow[rr] & & \bullet \arrow[r] & \bullet \arrow[rr] & & \bullet \end{tikzcd}$$

DOUBLE CATEGORY OF FUNCTORS & COFUNCTORS

A square of functors and cofunctors,

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,\varphi)}"'] & C \arrow[d, "{(g,\gamma)}"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

is compatible if:

(1) For all $a \in A$, $gha = kfa$

(2) For all $(a \in A, u : fa \to b \in B)$,

$$h\varphi(a,u) = \gamma(ha, ku)$$

<u>Note</u>: uniquely determined by boundary of square.

Let $\mathbb{C}\mathrm{of}$ be the double cat. whose:

- objects are categories
- horizontal morphisms are functors
- vertical morphisms are cofunctors
- cells are compatible squares

A delta lens is a cell in $\mathbb{C}\mathrm{of}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "{(f,\varphi)}"'] & B \arrow[d, "1_B"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

LENSES IN DOUBLE CATEGORIES

A **lens** $(f,\alpha,f') : A \nrightarrow B$ in a double category $\mathbb{D}$ is a cell of the form:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f'"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "1_B"] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

If $\mathbb{D}$ is **flat**, then lenses in $\mathbb{D}$ express **compatibility** between **horizontal** and **vertical** morphisms.

Lenses in $\mathbb{D}^v$

Cells in $\mathbb{D}$ $\rightsquigarrow$

$$
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "f'"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "1_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Lenses in $\mathbb{D}^h$

Cells in $\mathbb{D}$ $\rightsquigarrow$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1_B"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "f'"] \\
A \arrow[r, equal] & A
\end{tikzcd}
$$

Lenses in $\mathbb{D}^{vh}$

Cells in $\mathbb{D}$ $\rightsquigarrow$

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "1_A"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "f'"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

SPLIT EPIMORPHISMS AS LENSES

Let $\mathcal{C}$ be a category and $\mathbb{S}q(\mathcal{C})$ be the double category whose:

- objects are those of $\mathcal{C}$.
- horizontal and vertical morphisms are those in $\mathcal{C}$.
- cells are commutative squares in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

Lenses in $\mathbb{S}q(\mathcal{C})$ are boring; they are just morphisms in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d, "1_B"] \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$

Lenses in $\mathbb{S}q(\mathcal{C})^{v}$ are split epimorphisms:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "1_A"] \\ B \arrow[r, "\rho"'] & A \end{tikzcd}$$

ISOMORPHISMS & MONOMORPHISMS

Let $\mathcal{C}$ be a category and $\mathbb{P}\mathrm{b}(\mathcal{C})$ be the double category whose:

- objects are those of $\mathcal{C}$.
- horizontal and vertical morphisms are those in $\mathcal{C}$.
- cells are pullback squares in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

Lenses in $\mathbb{P}\mathrm{b}(\mathcal{C})$ are isomorphisms in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "1_B"] \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$

Lenses in $\mathbb{P}\mathrm{b}(\mathcal{C})^{vh}$ are monomorphisms:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

LENSES IN 2-CATEGORIES

Let $\mathcal{K}$ be a 2-category and let $\mathbb{Q}(\mathcal{K})$ be the double category whose:

- objects are those of $\mathcal{K}$.
- horizontal and vertical morphisms are those in $\mathcal{K}$.
- cells are given by diagrams in $\mathcal{K}$:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

Lenses in $\mathbb{Q}(\mathcal{K})$ are natural transformations:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "1_B"] \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$

Lenses in $\mathbb{Q}(\mathcal{K})^v$ and $\mathbb{Q}(\mathcal{K})^h$ are given by:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & A \arrow[d, "1_A"] \\ B \arrow[r, "\rho"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & C \arrow[d, "\rho"] \\ A \arrow[r, "1_A"'] & A \end{tikzcd}$$

LENSES IN BICATEGORIES

Let $\mathcal{K}$ be a bicategory and let $\mathbb{V}(\mathcal{K})$ be the double category whose:

- objects are those of $\mathcal{K}$.
- horizontal morphisms are identities
- vertical morphisms are those in $\mathcal{K}$.
- cells are given by 2-cells in $\mathcal{K}$:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & C \arrow[d, "g"] \\ B \arrow[r, equal] & D \end{tikzcd}$$

Lenses in $\mathbb{V}(\mathcal{K})$ are copointed endofunctors:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & A \arrow[d, "1_A"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

Lenses in $\mathbb{V}(\mathcal{K})^h$ are pointed endofunctors:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "1_A"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & A \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

SPLIT MULTI-VALUED FUNCTIONS

Let $\mathbb{S}\mathrm{pan}$ be the double category whose:

- objects are sets
- horizontal morphisms are functions
- vertical morphisms are spans
- cells are diagrams:

$$\begin{tikzcd}
A \arrow[r, "h"] & C \\
X \arrow[u, "f_1"] \arrow[d, "f_2"'] \arrow[r, "j"] & Y \arrow[u, "g_1"'] \arrow[d, "g_2"] \\
B \arrow[r, "k"'] & D
\end{tikzcd}$$

Lenses in $\mathbb{S}\mathrm{pan}^{vh}$ are split multi-valued functions:

$$\begin{tikzcd}
C \arrow[r, equal] & C \\
C \arrow[u, "1_C"] \arrow[d, "1_C"'] \arrow[r, "\sigma"] & Y \arrow[u, "g_1"'] \arrow[d, "g_2"] \\
C \arrow[r, "g_2\sigma"'] & D
\end{tikzcd}$$

<u>Fun fact</u>: The double category of split multi-valued functions classifies delta lenses! Delta lenses into $B \simeq$ lax double functors $\mathbb{V}(B) \longrightarrow \mathbb{S}\mathrm{Mult}$.

LENSES FROM COMONADS

Let $M : \mathcal{C} \to \mathcal{C}$ be a comonad and let $\mathbb{C}\mathrm{oKl}(M)$ be the double category whose:

- objects are those of $\mathcal{C}$
- horizontal morphisms are those of $\mathcal{C}$
- vertical morphisms are coKleisli maps
- cells are commutative squares in $\mathcal{C}$:

$$\begin{tikzcd}
MA \arrow[r, "Mh"] \arrow[d, "f"'] & MC \arrow[d, "g"] \\
B \arrow[r, "k"'] & D
\end{tikzcd}$$

A lens in $\mathbb{C}\mathrm{oKl}(M)^\vee$ is a morphism with an M-section:

$$\begin{tikzcd}
MA \arrow[r, equal, "M(1_A)"] \arrow[d, "\sigma"'] & MA \arrow[d, "\varepsilon_A"] \\
B \arrow[r, "\rho"'] & A
\end{tikzcd}$$

For the comonad $M \times (-) : \mathrm{Set} \to \mathrm{Set}$ we have:

$$\begin{tikzcd}
M \times B \arrow[r, equal] \arrow[d, "p"'] & M \times B \arrow[d, "\pi_A"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

CONSTRUCT YOUR OWN EXAMPLE

There are many possible examples that one can construct. Two options:

1. Pick your favourite double category $\mathbb{D}$ and compute lenses in it.
   - $\mathbb{P}\mathrm{rof}$
   - $\mathbb{B}\mathrm{iLens}(\mathcal{C},\otimes,I)$
   - $\mathbb{P}\mathrm{oly}$
   - $\mathbb{R}\mathrm{el}$
   - $\mathbb{O}\mathrm{ptic}(\mathcal{C},\otimes,I)$
   - $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$
   - $\mathbb{T}\mathrm{op}$
   - $\mathbb{A}\mathrm{rena}$
   - Free double cat on $H \longleftarrow C_0 \longrightarrow V$
2. Pick your favourite lens and try to construct it from a double category.

<u>Limitation</u>: The backwards components of your lens must be composable independently of the lens composition!

<u>Non-example</u>: Composition of bimorphic lenses $(A,A') \xrightarrow{(f,p)} (B,B') \xrightarrow{(g,q)} (C,C')$.

$$A \times C' \xrightarrow{\Delta \times 1} A \times A \times C' \xrightarrow{1 \times f \times 1} A \times B \times C' \xrightarrow{1 \times q} A \times B' \xrightarrow{p} A'$$

THE DOUBLE CATEGORY OF LENSES

Let $\mathbb{L}\mathrm{ens}(\mathbb{D})$ be the double category whose:

- objects are those of $\mathbb{D}$
- horizontal morphisms are those of $\mathbb{D}$
- vertical morphisms are lenses
- cells with boundary

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "{(f,\alpha,f')}"'] \arrow[dr, phantom, "\theta"] & C \arrow[d, "{(g,\beta,g')}"] \\
B \arrow[r, "k"'] & D
\end{tikzcd}$$

are cells θ in $\mathbb{D}$ such that:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\theta"] & C \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\beta"] & D \arrow[d, "1_D"] \\
B \arrow[r, "k"'] & D \arrow[r, equal] & D
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f'"'] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "1_k"] & D \arrow[d, "1_D"] \\
B \arrow[r, equal] & B \arrow[r, "k"'] & D
\end{tikzcd}$$

Composition of lenses is given by:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f'"'] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "g"] \arrow[d, "1_B"] \arrow[dr, phantom, "1_g"] & C \arrow[d, "1_C"] \\
B \arrow[r, equal] \arrow[d, "g'"'] \arrow[dr, phantom, "1_{g'}"] & B \arrow[r, "g"] \arrow[d, "g'"'] \arrow[dr, phantom, "\beta"] & C \arrow[d, "1_C"] \\
C \arrow[r, equal] & C \arrow[r, equal] & C
\end{tikzcd}$$

UNIVERSAL PROPERTY OF LENSES

A double category $\mathbb{D}$ is right-connected if for every vertical morphism $f : A \nrightarrow B$ there is a cell λ_f such that :

$$
\forall \quad
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"', "|" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "1_C", "|" marking] \\
B \arrow[r, "h"'] & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "\bar{f}"] \arrow[d, "f"', "|" marking] \arrow[dr, phantom, "\lambda_f"] & B \arrow[r, "h"] \arrow[d, "1_B", "|" marking] \arrow[dr, phantom, "1_h"] & C \arrow[d, "1_C", "|" marking] \\
B \arrow[r, equal] & B \arrow[r, "h"'] & C
\end{tikzcd}
$$

Let RCDbl be the full subcategory of Dbl_{unit} on r.c. double categories.

$$
\begin{tikzcd}
\mathsf{RCDbl} \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] & \mathsf{Dbl}_{unit} \arrow[l, shift right, "\mathbb{L}\mathrm{ens}(-)"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{C} \arrow[d, dashed, "\exists!"'] \arrow[dr] & \\
\mathbb{L}\mathrm{ens}(\mathbb{D}) \arrow[r] & \mathbb{D}
\end{tikzcd}
$$

right-connected

CANONICAL DOUBLE FUNCTORS

There are forgetful double functors:

$$\mathbb{D} \xleftarrow[\text{backwards component}]{\mathcal{L}} \mathbb{L}\mathrm{ens}(\mathbb{D}) \xrightarrow[\text{forwards component}]{\mathcal{U}} \mathbb{S}\mathrm{q}(\mathcal{D}_0)$$

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\theta"] & C \arrow[d, "g'"] & {} \arrow[d, phantom, "\leftsquigarrow"] & A \arrow[r, "h"] \arrow[d, "{(f,\alpha,f')}"'] \arrow[dr, phantom, "\theta"] & C \arrow[d, "{(g,\beta,g')}"] & {} \arrow[d, phantom, "\rightsquigarrow"] & A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "g"] \\
B \arrow[r, "k"'] & D & {} & B \arrow[r, "k"'] & D & {} & B \arrow[r, "k"'] & D
\end{tikzcd}$$

What if a lens was a backwards component with coalgebraic structure, or a forwards component with algebraic structure?

COALGEBRAIC PERSPECTIVE

A double category $\mathbb{D}$ is equivalent to an internal category in Cat:

$$
\begin{tikzcd}
\mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \arrow[r, "\mathrm{comp}"] & \mathcal{D}_1 \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathcal{D}_0 \arrow[l, "\mathrm{id}" description]
\end{tikzcd}
$$

where:

$\mathcal{D}_0$ - cat. of objects & hori. morph.

$\mathcal{D}_1$ - cat. of vert. morph. & cells

Assumptions:

(1) $\mathrm{cod} : \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ has a RARI

(2) Each fibre has finite products

Conjecture: The functor

$$\mathcal{L}ens(\mathbb{D}) \xrightarrow{\mathcal{L}_1} \mathcal{D}_1$$

is comonadic.

Idea: Given $f : A \nrightarrow B$, take product in fibre over B with $1_B : B \nrightarrow B$:

$$
\begin{tikzcd}
A \times_B B \arrow[r, "\pi_B"] \arrow[d, "f \times_B 1_B"', "\shortmid"{marking}] \arrow[dr, phantom, "\pi_{1_B}"] & B \arrow[d, "1_B", "\shortmid"{marking}] \\
B \arrow[r, equal] & B
\end{tikzcd}
\quad \text{cofree lens}
$$

• Conjecture is true for $\mathbb{D} = \mathbb{C}\mathrm{of}$.

ALGEBRAIC PERSPECTIVE

Every right-connected double category $\mathbb{C}$ admits a canonical double functor:

$$\mathbb{C} \xrightarrow{\mathcal{U}} \mathbb{S}\mathrm{q}(\mathcal{C}_0)$$

Bourke & Garner showed that

$$\mathcal{C}_1 \xrightarrow{\mathcal{U}_1} \mathcal{S}\mathrm{q}(\mathcal{C}_0) = \mathcal{C}_0^{\mathbf{2}}$$

is monadic if and only if $\mathcal{C}_0$ admits an algebraic weak factorisation system.

<u>Assumptions</u>:

(1) $\mathrm{dom} : \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ has a LARI

(2) $\mathrm{cod} : \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ is an opfibration

> <u>Conjecture</u>: The functor
>
> $$\mathcal{L}\mathrm{ens}(\mathbb{D}) \xrightarrow{\mathcal{U}_1} \mathcal{S}\mathrm{q}(\mathcal{D}_0)$$
>
> has a left adjoint.

- Conjecture is true for $\mathbb{D} = \mathbb{C}\mathrm{of}$.
- What about monadicity?

A COLLECTION OF PROPERTIES

The double category $\mathbb{L}\mathrm{ens}(\mathbb{D})$ has many nice properties:

If $\mathbb{D}$ is/has $\boxed{\qquad}$ then $\mathbb{L}\mathrm{ens}(\mathbb{D})$ is/has $\boxed{\qquad}$:

- Flat
- Companions
- Horizontal invariant
- Unit-pure with tabulators

- Conjoints in $\mathbb{L}\mathrm{ens}(\mathbb{D})$ arise precisely from horizontal isos.

- If $\mathbb{D} = \mathbb{C}\mathrm{of}$, then the codomain map $\mathrm{cod} \colon \mathcal{L}\mathrm{ens}(\mathbb{C}\mathrm{of}) \longrightarrow \mathrm{Cat}$ is a bifibration $\Rightarrow$ pullback and pushforward of delta lenses along functors exist.

SUMMARY & FUTURE WORK

- We defined lenses in a double category via certain cells:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f'"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "1_B", "\shortmid" marking] \\
B \arrow[r, equal] & B
\end{tikzcd}
\quad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "f'"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "1_B", "\shortmid" marking] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1_B"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "f'", "\shortmid" marking] \\
A \arrow[r, equal] & A
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "1_A"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "f'", "\shortmid" marking] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

- We considered several examples, including delta lenses.

- We constructed a double category $\mathbb{L}\mathrm{ens}(\mathbb{D})$ and studied its properties as the right-connected completion.

- <u>Future work</u>: More examples, extend framework, and prove (co)monadicity conjectures.