A general framework for cofunctors

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MOTIVATION

A cofunctor is a morphism of polynomial comonads on Set.

- A cofunctor is a kind of morphism of categories.
- How may we understand the implicit duality between functors and cofunctors, and what is the correct setting to study these notions together?
- How may we define the notion of cofunctor in the setting of internal categories, enriched categories, multicategories, etc.?
- Spoiler: Using morphisms of monads & double categories!

OUTLINE OF THE TALK

1. History and the definition of cofunctors.

2. The double category of cofunctors and its properties.

3. Cofunctors as monad morphisms including examples for monads in the double categories:
   - $\mathbb{S}\mathrm{pan}(\mathcal{C})$ and $\mathbb{S}\mathrm{pan}(\mathcal{C}, T)$
   - $\mathbb{M}\mathrm{at}(\mathcal{V})$
   - $\mathbb{C}\mathrm{omod}(\mathcal{V})$

4. Applications to generalised (delta) lenses and future work.

Part 1:

HISTORY AND THE DEFINITION OF COFUNCTORS

HISTORY & CONTEXT

1993: Higgins and Mackenzie introduce comorphisms between Lie groupoids (groupoids internal to cat. of smooth manifolds).

Definition 5·12. Let Ω and Ψ be Lie groupoids on bases M and N respectively. A *comorphism of Lie groupoids* $(\phi;f):(\Psi,N)\multimap(\Omega,M)$ consists of a smooth map $f:M\to N$, an action of Ψ on f, and a base-preserving morphism $\phi:\Psi\ltimes f\to\Omega$ over M.

Alternatively, we could work with a general decomposition; that is, a pair of morphisms

$$(\psi,f):(\Psi',M)\to(\Psi,N),\quad (\phi',\mathrm{id}):(\Psi',M)\to(\Omega,M)$$

such that ψ is an action morphism over f, and ϕ' is base-preserving.

1997: Aguiar defines a 2-category of categories, cofunctors, and transformations internal to a monoidal category $\mathcal{S}$.

First, in the case $\mathcal{S} = \mathcal{S}ets$, we may describe functors $f:\mathfrak{C}\to\mathfrak{D}$ and cofunctors $\varphi:\mathfrak{C}\to\mathfrak{D}$ through pictures as follows.

$$\begin{tikzcd} \mathfrak{C} \arrow[d, "f"'] \\ \mathfrak{D} \end{tikzcd}
\qquad
\begin{tikzcd} x \arrow[rr, "a"] \arrow[d, mapsto, "f_0"'] & {} \arrow[d, mapsto, "f_1"] & x' \arrow[d, mapsto, "f_0"] \\ f_0(x) \arrow[rr, "f_1(a)"'] & {} & f_0(x') \end{tikzcd}
\qquad
\begin{tikzcd} \mathfrak{D} \\ \mathfrak{C} \arrow[u, "\varphi"] \end{tikzcd}
\qquad
\begin{tikzcd} y \arrow[rr, "\varphi_1(a,y)"] \arrow[d, mapsto, "\varphi_0"'] & {} & y' \arrow[d, mapsto, "\varphi_0"] \\ \varphi_0(y) \arrow[rr, "a"'] & {} \arrow[u, mapsto, "\varphi_1"'] & \varphi_0(y') \end{tikzcd}$$

HISTORY & CONTEXT

2016/2017: Ahman & Uustalu show that polynomial comonads are categories, and the morphisms are cofunctors.

A *cofunctor* between small categories $(S, \bar{P}, \mathsf{src}, \mathsf{tgt}, \mathsf{id}, ;)$ and $(S_0, \bar{P}_0, \mathsf{src}_0, \mathsf{tgt}_0, \mathsf{id}_0, ;_0)$ is given by two maps $t : S_0 \to S$ (the object map) and $\bar{q} : (\Sigma s_0 : S_0 . \Sigma p : \bar{P} . t\, s_0 = \mathsf{src}\, p) \to \bar{P}_0$ (the morphism map) satisfying $\mathsf{src}_0\, (\bar{q}\,(s_0, p)) = s_0$ and

$$t\,(\mathsf{tgt}_0\,(\bar{q}\,(s_0, p))) = \mathsf{tgt}\, p$$
$$\mathsf{id}_{0\, s_0} = \bar{q}\,(s_0, \mathsf{id}_{t\, s_0})$$
$$\bar{q}\,(s_0, p) ;_0 \bar{q}\,(\mathsf{tgt}_0\,(\bar{q}\,(s_0, p)), p') = \bar{q}\,(s_0, p\, ; p')$$

2019+: Continued progress towards understanding cofunctors by Garner, Cockett, Paré, Spivak, Niu, Di Meglio, ...

The textbook Polynomial Functors: A General Theory of Interaction (Spivak & Niu) contains a detailed account of cofunctors via polynomials.

COFUNCTORS BETWEEN CATEGORIES

A cofunctor $(f, \varphi) : A \nrightarrow B$ consists of an object assignment,

$$f : \mathrm{Obj}(A) \longrightarrow \mathrm{Obj}(B)$$

and a lifting operation,

$$\begin{tikzcd}
A \arrow[d, "{(f,\varphi)}"', "\shortmid" marking] & a \arrow[r, "{\varphi(a,u)}"] \arrow[d, phantom, "\vdots"] & a' \arrow[d, phantom, "\vdots"] \\
B & fa \arrow[r, "u"] & b = fa'
\end{tikzcd}$$

where $a' = \mathrm{cod}(\varphi(a,u))$, such that:

(1) $\varphi(a, 1_{fa}) = 1_a$

(2) $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a, u)$

Composition of cofunctors:

$$\begin{tikzcd}
A \arrow[d, "{(f,\varphi)}"', "\shortmid" marking] & a \arrow[r, "{\varphi(a,\gamma(fa,u))}"] \arrow[d, phantom, "\vdots"] & a' \arrow[d, phantom, "\vdots"] \\
B \arrow[d, "{(g,\gamma)}"', "\shortmid" marking] & fa \arrow[r, "{\gamma(fa,u)}"] \arrow[d, phantom, "\vdots"] & b \arrow[d, phantom, "\vdots"] \\
C & gfa \arrow[r, "u"] & c
\end{tikzcd}$$

Let $\mathcal{C}at^{\#}$ denote the category of (small) categories and cofunctors.

EXAMPLES

- Every split opfibration has an underlying cofunctor: its splitting.

- A cofunctor between codiscrete cats. is a very well behaved lens.

$$\begin{tikzcd} A \arrow[d, "\text{cofunctor}"', "/" marking] & \bullet \arrow[r, "\varphi(u)"] \arrow[d, phantom, "\vdots"] & \bullet \arrow[d, phantom, "\vdots"] & A \\ B & \bullet \arrow[r, "u"] & \bullet & B \arrow[u, "\text{monoid homo.}"'] \end{tikzcd}$$

- There is a fully faithful functor:

$$\mathrm{Mon}^{op} \hookrightarrow \mathrm{Cat}^{\#}$$

- More generally, every bijective-on-objects (b.o.o.) functor $B \to A$ yields a b.o.o. cofunctor $A \nrightarrow B$.

- Every discrete opfibration $A \to B$ yields a cofunctor $A \nrightarrow B$

- A cofunctor $A \overset{\tau}{\nrightarrow} (\mathbb{N}, +, 0)$ is a choice of morphism out of each object in A.

$$a \in A \longmapsto \tau(a,1) : a \longrightarrow \bullet$$

COFUNCTORS AS SPANS OF FUNCTORS

- There is an O.F.S. on $\mathcal{C}at^{\#}$ factoring each cofunctor into a b.o.o. cofunctor followed by a discrete opfibration.

$$\begin{tikzcd}
A \arrow[d, "\text{b.o.o.}", "|"'] \arrow[dd, bend right=60, "f"', "|"] & a \arrow[rr, "\varphi(a,u)"] & & a' & A \\
\Lambda \arrow[d, "\text{d.opf.}", "|"'] & a \arrow[rr, "{(a,u)}"] & & a' & \Lambda \arrow[u, Rightarrow, "\varphi"'] \arrow[d, Rightarrow, "\bar{f}"] \\
B & fa \arrow[rr, "u"] & & b & B
\end{tikzcd}$$

- Cofunctor $\simeq$ certain span of functors.

- There is a special universal property: Given a cofunctor $(f,\varphi): A \nrightarrow B$ and a span of functors,

$$\begin{tikzcd}
A & C \arrow[l, "g"'] \arrow[r, "h"] & B
\end{tikzcd}$$

such that $\forall c \in C$, $hc = fgc$, and $\forall u: c \to c' \in C$, $gu = \varphi(gc, hu)$, there is a unique functor:

$$\begin{tikzcd}
& C \arrow[dl, "g"'] \arrow[dr, "h"] \arrow[dd, dashed, "\exists!"] & \\
A & & B \\
& \Lambda \arrow[ul, "\varphi"] \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

Part 2:

THE DOUBLE CATEGORY OF COFUNCTORS AND ITS PROPERTIES

DOUBLE CATEGORIES

- A double category $\mathbb{D}$ consists of:
  - objects: $A, B, C, D, \ldots$
  - horizontal morphisms: $\bullet \longrightarrow \bullet$
  - vertical morphisms: $\bullet \,\not\longrightarrow\, \bullet$
  - cells:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\theta"] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

where vertical composition is associative up to isomorphism.

- A double category is a pseudo internal category in CAT,

$$\begin{tikzcd} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\mathrm{comp}"] & \mathbb{D}_1 \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathbb{D}_0 \arrow[l, "\mathrm{id}"'] \end{tikzcd}$$

where:

$\mathbb{D}_0$ - cat. of objects & hori. morph.

$\mathbb{D}_1$ - cat. of vert. morph. & cells

- A double category is flat if its cells are dertermined by their boundary morphisms.

PROPERTIES OF DOUBLE CATEGORIES

A horizontal morphism $f\colon A \to B$ has:

- A **companion** $f_*\colon A \nrightarrow B$ if there are cells

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f_*"', "\shortmid" marking] \arrow[dr, phantom, "\heartsuit"] & B \arrow[d, "1", "\shortmid" marking] \\ B \arrow[r, equal] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "1"', "\shortmid" marking] \arrow[dr, phantom, "\diamondsuit"] & A \arrow[d, "f_*", "\shortmid" marking] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

satisfying the equations:

$$\diamondsuit \,|\, \heartsuit = 1_f \qquad \frac{\diamondsuit}{\heartsuit} = 1_{f_*}$$

- A **conjoint** $f^*\colon B \nrightarrow A$ if there are cells

$$\begin{tikzcd} B \arrow[r, equal] \arrow[d, "f^*"', "\shortmid" marking] \arrow[dr, phantom, "\clubsuit"] & B \arrow[d, "1", "\shortmid" marking] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "1"', "\shortmid" marking] \arrow[dr, phantom, "\spadesuit"] & B \arrow[d, "f^*", "\shortmid" marking] \\ A \arrow[r, equal] & A \end{tikzcd}$$

satisfying similar equations.

- The **tabulator** of $f\colon A \nrightarrow B$ is

$$\begin{tikzcd} Tf \arrow[r, "\pi_A"] \arrow[d, "1"', "\shortmid" marking] \arrow[dr, phantom, "\pi_f"] & A \arrow[d, "f", "\shortmid" marking] \\ Tf \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

which is "universal".

- $\mathbb{D}$ has **strong tabulators** if

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "(\pi_A)^*"', "\shortmid" marking] & A \arrow[dd, "f", "\shortmid" marking] \\ Tf \arrow[d, "(\pi_B)_*"', "\shortmid" marking] \arrow[r, phantom, "\begin{array}{c}\clubsuit \\ \hline \pi_f \\ \hline \heartsuit\end{array}"] & {} \\ B \arrow[r, equal] & B \end{tikzcd}$$

is invertible for all $f\colon A \nrightarrow B$.

FUNCTORS & COFUNCTORS

- A square of functors and cofunctors,

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "{(f,\varphi)}"'] & C \arrow[d, "{(g,\gamma)}"] \\
B \arrow[r, "k"'] & D
\end{tikzcd}
$$

is compatible if:

(1) For all $a \in A$, $gha = kfa$

(2) For all $(a \in A, u : fa \to b \in B)$,

$$h\varphi(a,u) = \gamma(ha, ku)$$

- Let $\mathcal{C}of$ denote the category of cofunctors and compatible squares.

- A compatible square $\simeq$ diagram of functors:

$$
\begin{tikzcd}
A \arrow[r, "h"] & C \\
\Lambda_{f,\varphi} \arrow[u, "\varphi"] \arrow[r, "\exists!"] \arrow[d, "\bar{f}"'] & \Lambda_{g,\gamma} \arrow[u, "\gamma"'] \arrow[d, "\bar{g}"] \\
B \arrow[r, "k"'] & D
\end{tikzcd}
$$

- Let $\mathbb{C}of$ be the double cat:

$$
\begin{tikzcd}
\bullet \arrow[r, "\mathrm{comp}"] & \mathcal{C}of \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathcal{C}at \arrow[l, "\mathrm{id}" description]
\end{tikzcd}
$$

PROPERTIES OF COFUNCTORS

The flat double category $\mathbb{C}\mathrm{of}$ of categories, functors, cofunctors, and compatible squares has many nice properties:

- Companions = discrete opfibrations
- Conjoints = bijective-on-objects func.
- Strong tabulators exist:

$$\begin{tikzcd} \Lambda \arrow[r, "\varphi"] \arrow[d, "1"'] & A \arrow[d, "{(f,\varphi)}"] \\ \Lambda \arrow[r, "\overline{f}"'] & B \end{tikzcd}$$

- Span representable via a fully faithful (pseudo) double functor:

$$\mathbb{C}\mathrm{of} \xrightarrow{\ \Lambda\ } \mathbb{S}\mathrm{pan}(\mathcal{C}\mathrm{at})$$

- The internal perspective:

$$\begin{tikzcd}[row sep=small] \mathcal{C}\mathrm{of} \arrow[r, bend left=60, leftarrow] \arrow[r, bend left=30, "\mathrm{dom}"] \arrow[r, hookleftarrow, "\mathrm{id}"] \arrow[r, bend right=30, "\mathrm{cod}"'] \arrow[r, bend right=60, leftarrow] & \mathcal{C}\mathrm{at} \end{tikzcd}$$

- Codomain map is a bifibration, whose fibres $\mathcal{C}\mathrm{of}_B$ have finite limits.

What is the general framework?

Part 3:

Cofunctors as monad morphisms with examples

MONADS IN DOUBLE CATEGORIES

A **vertical monad** (A, t, η, μ) in $\mathbb{D}$ consists of a vertical endomorphism $t : A \to\!\!\!\!\!\!\!+\; A$ and globular cells,

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "1_A"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\eta}{\Rightarrow}"] & A \arrow[d, "t", "\shortmid"' marking] \\ A \arrow[r, equal] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "t"', "\shortmid" marking] & A \arrow[dd, "t", "\shortmid"' marking] \\ A \arrow[d, "t"', "\shortmid" marking] \arrow[r, phantom, "\overset{\mu}{\Rightarrow}"] & \\ A \arrow[r, equal] & A \end{tikzcd}$$

satisfying the usual axioms.

A **monad map** $(f, \bar{f}) : (A, t) \longrightarrow (B, s)$ consists of a horizontal morphism $f : A \longrightarrow B$ and a cell,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "t"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\bar{f}}{\Rightarrow}"] & B \arrow[d, "s", "\shortmid"' marking] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

satisfying axioms using η and μ.

<u>Example</u> ($\mathbb{D} = \mathbb{S}\mathrm{pan}$): monads and monad maps are categories and functors.

MONAD RETROMAPS

A **vertical monad map** $(f,\varphi):(A,t) \nrightarrow (B,s)$ consists of a vertical morphism $f: A \nrightarrow B$ and a cell,

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"', "\shortmid" marking] & A \arrow[d, "t", "\shortmid" marking] \\
B \arrow[d, "s"', "\shortmid" marking] \arrow[r, phantom, "\varphi"] & A \arrow[d, "f", "\shortmid" marking] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

satisfying axioms using η and μ.

$\mathbb{M}\mathrm{nd}(\mathbb{D})$: double cat of monads, monad maps, & vertical monad maps.

What if $f: A \nrightarrow B$ is a **conjoint** or **companion**?

$\Rightarrow$ For $\mathbb{D}$ with conjoints: $(f^*, \varphi) \simeq$ **monad map**

$\Rightarrow$ For $\mathbb{D}$ with companions: $(f_*, \varphi) :=$ **monad retromap**

Define full double subcategory

$$\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D}) \hookrightarrow \mathbb{M}\mathrm{nd}(\mathbb{D})$$

on the monad retromaps. For $\mathbb{D} = \mathbb{S}\mathrm{pan}$, $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D}) \simeq \mathbb{C}\mathrm{of}$.

INTERNAL COFUNCTORS

- For $\mathcal{C}$ with pullbacks, let $\mathbb{S}\mathrm{pan}(\mathcal{C})$ be the double category of spans in $\mathcal{C}$.
- For $T : \mathcal{C} \to \mathcal{C}$ a pullback-preserving cartesian monad, let $\mathbb{S}\mathrm{pan}(\mathcal{C}, T)$ be the double cat. of T-spans.

$$TA \longleftarrow X \longrightarrow B$$

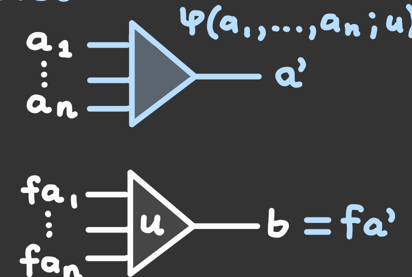
- Example: Let T be the List monad on Set.

$\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan}(\mathcal{C}))$ consists of internal cats., internal functors, and internal cofunctors.

Example: comorphisms of Lie gpds.

For $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathrm{Set}, \mathrm{List})$, monad retromaps are cofunctors between multicategories.

$$\begin{array}{c} A \\ {\scriptstyle (f,\varphi)}\ \updownarrow \\ B \end{array}$$



ENRICHED COFUNCTORS

- For a distributitive monoidal cat. $(\mathcal{V}, \otimes, I)$, let $\mathbb{M}\mathrm{at}(\mathcal{V})$ be the double category whose:
  - objects and horizontal morphisms are sets and functions
  - vertical morphisms are $\mathcal{V}$-matrices
    $$M : A \times B \longrightarrow \mathcal{V}$$
  - cells are natural transformations:
$$\begin{tikzcd} A \times B \arrow[d, "h \times k"'] \arrow[dr, "M"] \arrow[dr, phantom, shift right=3ex, "\Downarrow\alpha"] & \\ C \times D \arrow[r, "N"'] & \mathcal{V} \end{tikzcd}$$

For $\mathbb{D} = \mathbb{M}\mathrm{at}(\mathcal{V})$, monad retromaps are enriched cofunctors. In detail, $(F, \varphi) : \mathcal{A} \longrightarrow \mathcal{B}$ consists of:

- A function $F : \mathcal{A}_0 \longrightarrow \mathcal{B}_0$.
- For each $a \in \mathcal{A}$ and $b \in \mathcal{B}$, a lifting operation in $\mathcal{V}$,
$$\mathcal{B}(Fa, b) \xrightarrow{\varphi_{a,b}} \sum_{x \in X} \mathcal{A}(a, x)$$

where $X = F^{-1}(b)$ is the fibre of F over b.

<u>Example</u>: weighted cofunctors, $\mathcal{V} = \mathrm{wSet}$.

AGUIAR'S COFUNCTORS

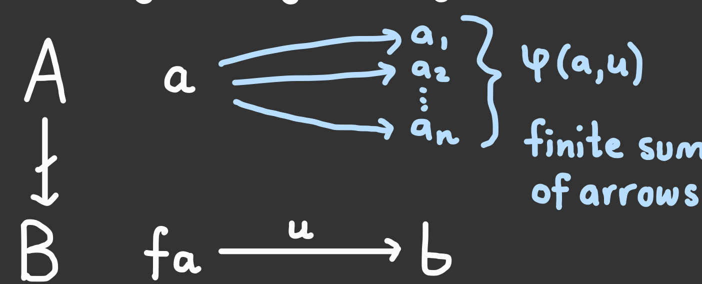
Consider a monoidal cat. $(\mathcal{V}, \otimes, I)$ with equalisers which are preserved by the functors $A \otimes (-) \otimes B : \mathcal{V} \longrightarrow \mathcal{V}$.

Let $\mathbb{C}\mathrm{omod}(\mathcal{V})$ be the double category whose:

- objects are comonoids
- horizontal morphisms are comonoid homomorphisms
- vertical morphisms are two-sided comodules

For $\mathbb{D} = \mathbb{C}\mathrm{omod}(\mathcal{V})$, monad retromaps are internal cofunctors in the sense of Aguiar.

<u>Example</u>: When $\mathcal{V} = \mathrm{Vec}_k$, we have cofunctors between linear categories. Projecting to ordinary categories yields:



PROPERTIES OF MONAD RETROMAPS

The double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ inherits several properties from a unit-pure $\mathbb{D}$ with companions:

- It is flat.
- Monad map $(f, \bar{f}) : (A, t) \longrightarrow (B, s)$ has:
  - conjoint iff $f : A \to B$ is invertible.
  - companion iff cell below is an iso:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "t\,;\,f_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\frac{\frac{\diamond}{\bar{f}}}{\heartsuit}"] & A \arrow[d, "f_*\,;\,s", "\shortmid"{marking}] \\ B \arrow[r, equal] & B \end{tikzcd}$$

- On the other hand, many properties are difficult to prove at the level of $\mathbb{D}$.
- <u>Conjecture</u>: $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ has tabulators if $\mathbb{D}$ is strongly span representable.
- In specific cases of $\mathbb{D}$:
  - $\mathbb{S}\mathrm{pan}(\mathcal{C})$ ✓
  - $\mathbb{M}\mathrm{at}(\mathcal{V})$ ✓ if $\mathcal{V}$ is extensive
  - $\mathbb{C}\mathrm{omod}(\mathcal{V})$ unknown

Part 4:

APPLICATIONS TO DELTA LENSES AND FUTURE WORK

A FRAMEWORK FOR GENERALISED LENSES

- A (delta) lens is a compatible functor and cofunctor pair.

- Lens $A \to B \simeq$ cell in $\mathbb{C}\mathrm{of}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "(f,\varphi)"'] & B \arrow[d, "1_B"] \\ B \arrow[r, equal] & B \end{tikzcd} \qquad f\varphi(a,u) = u$$

- A double cat. $\mathbb{D}$ is right-connected if $\mathrm{id}\colon \mathcal{D}_0 \to \mathcal{D}_1$ is right adjoint to $\mathrm{cod}\colon \mathcal{D}_1 \to \mathcal{D}_0$.

$$\mathbb{D} \longrightarrow \boxed{\text{Right-connected completion}} \longrightarrow \Gamma(\mathbb{D})$$

- $\mathbb{L}\mathrm{ens} = \Gamma(\mathbb{C}\mathrm{of})$ and $\mathcal{L}\mathrm{ens} \longrightarrow \mathcal{C}\mathrm{of}$ is comonadic.

$$\begin{tikzcd} LA \arrow[d] & a \arrow[r, "{(u,\varphi(a,u))}"] \arrow[d, phantom, "\vdots"] & a' \arrow[d, phantom, "\vdots"] \\ B & fa \arrow[r, "u"] & b \end{tikzcd}$$

cofree lens on (f,φ)

- $\mathbb{L}\mathrm{ens}(\mathbb{D}) = \Gamma(\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D}))$ yields lenses between monads.

CURRENT & FUTURE RESEARCH

- What other properties of $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ may be inherited from $\mathbb{D}$?

- Are there more interesting examples of cofunctors for some $\mathbb{D}$?

- Can all properties of cofunctors be understood via representation as spans of functors?

- Can we link back to polynomials?

- Is there a fibred approach to cofunctors?

$$\begin{tikzcd} \mathrm{DOpf}_B \arrow[r, hook] \arrow[d, "\mathrm{iso}"'] & \mathrm{Cof}_B \arrow[d, dashed, "\mathrm{iso}"] \\ {[B, \mathrm{Set}]} \arrow[r, dashed] & \boxed{?} \end{tikzcd}$$

In other words, is there a double cat. $\mathbb{D}$ such that

$$\frac{\text{cofunctors into } B}{\text{(lax) double functors } B \longrightarrow \mathbb{D}}$$

SUMMARY

- Introduced a general framework for the theory of cofunctors via the double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ of monads, monad maps, and monad retromaps in a double category $\mathbb{D}$ with companions.

- Functors and cofunctors are dual and can be studied together:

$\mathbb{C}\mathrm{of} \simeq \mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ for $\mathbb{D} = \mathbb{S}\mathrm{pan}$

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{(f,\varphi)}"'] & C \arrow[d, "{(g,\gamma)}"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

$$gha = kfa$$
$$h\varphi(a,u) = \gamma(ha, ku)$$

- Can use this setting to generalise cofunctors (internal, enriched) and study delta lenses.

2 1 A BETTER DOUBLE CATEGORY OF COFUNCTORS