Enriched lenses

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Joint work with Matthew Di Meglio

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MOTIVATION

Lifting couplings in Wasserstein spaces

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This paper makes mathematically precise the idea that conditional probabilities are analogous to *path liftings* in geometry.

The idea of lifting is modelled in terms of the category-theoretic concept of a *lens*, which can be interpreted as a consistent choice of arrow liftings. The category we study is the one of probability measures over a given standard Borel space, with morphisms given by the couplings, or transport plans.

The geometrical picture is even more apparent once we equip the arrows of the category with *weights*, which one can interpret as "lengths" or "costs", forming a so-called *weighted category*, which unifies several concepts of category theory and metric geometry. Indeed, we show that the weighted version of a lens is tightly connected to the notion of *submetry* in geometry.

Every weighted category gives rise to a pseudo-quasimetric space via optimization over the arrows. In particular, Wasserstein spaces can be obtained from the weighted categories of probability measures and their couplings, with the weight of a coupling given by its cost. In this case, conditionals allow one to form weighted lenses, which one can interpret as "lifting transport plans, while preserving their cost".

- Nov 2018 - Generalise lenses to the internal category setting.
- 2021 - Mathew di Meglio and I develop enriched lenses, but lack compelling examples.
- Earlier this month, Perrone posts a paper introducing lenses between weighted categories.

OVERVIEW OF THE TALK

1. Background
2. Defining enriched lenses
3. Weighted lenses

Warning: Notation may be globally inconsistent

DOUBLE CATEGORIES & COMPANIONS

A **double category** $\mathbb{D}$ consists of:

- objects $A, B, C, \ldots$
- horizontal (tight) morphisms
- vertical (loose) morphisms
- cells

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"', "|" marking] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "v", "|" marking] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
$$

where vertical composition is associative up to isomorphism.

A horizontal arrow $f : A \to B$ has a **companion** $f_* : A \nrightarrow B$ if there are binding cells,

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & A \arrow[d, "f_*", "|" marking] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f_*"', "|" marking] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

such that

$$
\alpha \mid \beta = 1_f \qquad \frac{\alpha}{\beta} = 1_{f_*}
$$

PARÉ'S RETROCELLS

- A retrocell in $\mathbb{D}$ is a cell of the form:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{\alpha}{\Leftarrow}"] & B \arrow[d, "v"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f_*"'] & A \arrow[d, "u"] \\
B \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[d, "v"'] & C \arrow[d, "g_*"] \\
D \arrow[r, equal] & D
\end{tikzcd}
$$

- Like ordinary cells, retrocells can be composed vertically and horizontally.

- If $\mathbb{D}$ has all companions, we can construct a double category $\mathbb{D}^{ret}$.

CELLS VS. RETROCELLS

In a double category $\mathbb{D}$ with companions:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"', "|" marking] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "v", "|" marking] \\ C \arrow[r, "g"'] & D \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"', "|" marking] \arrow[dr, phantom, "\overset{\beta}{\Leftarrow}"] & B \arrow[d, "v", "|" marking] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

cells $\qquad \wr \qquad\qquad \wr \qquad$ retrocells

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "u"', "|" marking] & A \arrow[d, "f_*", "|" marking] \\ C \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[d, "g_*"', "|" marking] & B \arrow[d, "v", "|" marking] \\ D \arrow[r, equal] & D \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "f_*"', "|" marking] & A \arrow[d, "u", "|" marking] \\ B \arrow[r, phantom, "\overset{\beta}{\Rightarrow}"] \arrow[d, "v"', "|" marking] & C \arrow[d, "g_*", "|" marking] \\ D \arrow[r, equal] & D \end{tikzcd}$$

DISTRIBUTIVE MONOIDAL CATEGORIES

- A monoidal category is a category $\mathcal{C}$ equipped with functors,

$$\underset{\text{tensor product}}{\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}} \qquad \underset{\text{unit}}{I : * \longrightarrow \mathcal{C}}$$

and natural isomorphisms with components,

$$(X \otimes Y) \otimes Z \xrightarrow{\alpha} X \otimes (Y \otimes Z) \qquad I \otimes X \xrightarrow{\lambda} X \xleftarrow{\rho} X \otimes I$$

which satisfy the triangle and pentagon identities.

- A monoidal category is distributive if it has coproducts such that:

$$X \otimes \left(\sum_i Y_i\right) \cong \sum_i (X \otimes Y_i) \qquad \left(\sum_i X_i\right) \otimes Y \cong \sum_i (X_i \otimes Y)$$

WEIGHTED SETS

- Consider the set $[0,\infty]$ as a category with morphisms $\lambda \geqslant \mu$.
- Has several monoidal structures, including $+ : [0,\infty] \times [0,\infty] \longrightarrow [0,\infty]$.
- The category of **weighted sets** is defined by $\omega\mathrm{Set} := \mathrm{Fam}([0,\infty])$.

$$\begin{tikzcd} X \arrow[d, "|\cdot|"'] \\ {[0,\infty]} \end{tikzcd}$$

objects

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "|\cdot|"'] \arrow[rr, phantom, shift right=3ex, "\Rightarrow"] & & Y \arrow[dl, "|\cdot|"] \\ & {[0,\infty]} & \end{tikzcd}$$

morphisms

$|fx| \leqslant |x|$

$$\begin{tikzcd} X \times Y \arrow[rr] \arrow[dr, "|\cdot|+|\cdot|"'] & & {[0,\infty] \times [0,\infty]} \arrow[dl, "+"] \\ & {[0,\infty]} & \end{tikzcd}$$

tensor product +

$$\begin{tikzcd} 1 \arrow[d, "0"] \\ {[0,\infty]} \end{tikzcd}$$

unit

- The category $\omega\mathrm{Set}$ has coproducts $X \sqcup Y \longrightarrow [0,\infty]$ by construction and is a distributive monoidal category with respect to addition.

THE DOUBLE CATEGORY OF MATRICES

Given a distributive monoidal category $(\mathcal{V}, \otimes, I)$, there is a double category $\mathcal{V}\text{-}\mathbb{M}\mathrm{at}$ whose:

- objects are sets and horizontal morphisms are functions;
- vertical morphisms $M : A \nrightarrow B$ are functors $M : A \times B \longrightarrow \mathcal{V}$;
- cells given by natural transformations:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "M"', "|" marking] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "N", "|" marking] \\ C \arrow[r, "g"'] & D \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \times C \arrow[dd, "f \times g"'] \arrow[drr, "M"] & & \\ & \Downarrow \alpha & \mathcal{V} \\ B \times D \arrow[urr, "N"'] & & \end{tikzcd}$$

THE DOUBLE CATEGORY OF MATRICES

- **Vertical identity** morphisms are:

$$A \times A \xrightarrow{\delta} \mathcal{V}$$

$$(x, y) \longmapsto \delta_{x,y} = \begin{cases} I & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

- **Vertical composition** of $M : A \nrightarrow B$ with $N : B \nrightarrow C$ is given by:

$$A \times C \longrightarrow \mathcal{V}$$

$$(a, c) \longmapsto \sum_{b \in B} M(a,b) \otimes N(b,c)$$

- A function $f : A \to B$ has **companion**:

$$A \times B \xrightarrow{f_*} \mathcal{V}$$

$$(x, y) \longmapsto \delta_{fx,y} = \begin{cases} I, & fx = y \\ 0, & fx \neq y \end{cases}$$

- **Retrocells** are given by:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "M"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\alpha}{\Leftarrow}"] & B \arrow[d, "N", "\shortmid" marking] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

$$\alpha_{a,d} : N(fa, d) \longrightarrow \sum_{c \in g^{-1}(d)} M(a,c)$$

MONADS & ENRICHED CATEGORIES

A **vertical monad** (A, t, η, μ) in $\mathbb{D}$ consists of a vertical endomorphism $t : A \relbar\joinrel\mapstochar\joinrel\rightarrow A$ and special cells,

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "1_A"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\eta}{\Rightarrow}"] & A \arrow[d, "t", "\shortmid" marking] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "t"', "\shortmid" marking] & A \arrow[dd, "t", "\shortmid" marking] \\
A \arrow[d, "t"', "\shortmid" marking] \arrow[r, phantom, "\overset{\mu}{\Rightarrow}"] & {} \\
A \arrow[r, equal] & A
\end{tikzcd}$$

satisfying the usual axioms.

A vertical monad in $\mathcal{V}$-$\mathbb{M}\mathrm{at}$ is a **$\mathcal{V}$-category**

- A set $\mathcal{C}_0$ of objects;
- A functor $\mathcal{C} : \mathcal{C}_0 \times \mathcal{C}_0 \longrightarrow \mathcal{V}$ specifying hom-objects $\mathcal{C}(x, y)$;
- A unit map $I \xrightarrow{\eta} \mathcal{C}(x, x)$ for each $x \in \mathcal{C}_0$.
- A multiplication map
$$\mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \xrightarrow{\mu} \mathcal{C}(x, z)$$
for each $x, y, z \in \mathcal{C}_0$.

MONAD MAPS & ENRICHED FUNCTORS

A monad map $(f, \bar{f}) : (A, t) \longrightarrow (B, s)$ consists of a horizontal arrow $f : A \longrightarrow B$ and a cell,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\bar{f}}{\Rightarrow}"] & B \arrow[d, "s"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

satisfying axioms which ensure compatibility with the unit and multiplication cells.

A monad map in $\mathcal{V}\text{-}\mathbb{M}\mathrm{at}$ is an enriched functor $F : \mathcal{C} \longrightarrow \mathcal{D}$.

- A function $F : \mathcal{C}_0 \longrightarrow \mathcal{D}_0$.
- A morphism $F_{x,y} : \mathcal{C}(x,y) \longrightarrow \mathcal{D}(Fx, Fy)$ for each $x, y \in \mathcal{C}_0$.
- Satisfying the axioms:

$$F_{x,x} \circ \eta = \eta$$

$$F_{x,z} \circ \mu = \mu \circ (F_{x,y} \otimes F_{y,z})$$

MONAD RETRO MAPS

A **monad retro map** $(f,\varphi):(A,t)\longrightarrow(B,s)$ consists of a horizontal arrow $f:A\longrightarrow B$ and a retrocell,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\overset{\varphi}{\Leftarrow}"] & B \arrow[d, "s"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

satisfying axioms which ensure compatibility with the unit and multiplication cells.

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f_*"'] & A \arrow[r, equal] \arrow[d, "f_*"'] \arrow[ddr, phantom, "\overset{\varphi}{\Rightarrow}"] & A \arrow[d, "t"] \\
B \arrow[r, equal] \arrow[d, "1_B"'] \arrow[dr, phantom, "\overset{\eta}{\Rightarrow}"] & B \arrow[d, "s"'] & A \arrow[d, "f_*"] \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}$$

unit law

$$=$$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "1_A"'] \arrow[dr, phantom, "\overset{\eta}{\Rightarrow}"] & A \arrow[d, "t"] \\
A \arrow[r, equal] \arrow[d, "f_*"'] & A \arrow[d, "f_*"] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

MONAD RETRO MAPS

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f_*"'] & A \arrow[r, equal] \arrow[d, "t"'] & A \arrow[r, equal] \arrow[d, "t"'] & A \arrow[dd, "t"] & & A \arrow[r, equal] \arrow[d, "f_*"'] & A \arrow[r, equal] \arrow[d, "f_*"'] & A \arrow[dd, "t"] \\
B \arrow[d, "s"'] \arrow[r, phantom, "\overset{\varphi}{\Rightarrow}"] & A \arrow[r, equal] \arrow[d, "f_*"'] & A \arrow[d, "t"] \arrow[r, phantom, "\overset{\mu}{\Rightarrow}"] & & = & B \arrow[r, equal] \arrow[d, "s"'] & B \arrow[dd, "s"'] \arrow[r, phantom, "\overset{\varphi}{\Rightarrow}"] & \\
B \arrow[r, equal] \arrow[d, "s"'] & B \arrow[d, "s"'] \arrow[r, phantom, "\overset{\varphi}{\Rightarrow}"] & A \arrow[r, equal] \arrow[d, "f_*"] & A \arrow[d, "f_*"] & & B \arrow[d, "s"'] \arrow[r, phantom, "\overset{\mu}{\Rightarrow}"] & & A \arrow[d, "f_*"] \\
B \arrow[r, equal] & B \arrow[r, equal] & B \arrow[r, equal] & B & & B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}
$$

multiplication law

ENRICHED COFUNCTORS

A monad retro map in $\mathcal{V}$-$\mathbb{M}$at is an enriched cofunctor $(F, \varphi) : \mathcal{C} \to \mathcal{D}$.

- A function $F : \mathcal{C}_0 \to \mathcal{D}_0$.
- A morphism for each $c \in \mathcal{C}_0$ and $d \in \mathcal{D}_0$ called a lifting operation,

$$\mathcal{D}(Fc, d) \xrightarrow{\varphi_{c,d}} \sum_{x \in X} \mathcal{C}(c, x)$$

where $X = F^{-1}(d)$ is the fibre of F over d.

$$\begin{tikzcd} I \arrow[r, "\eta"] \arrow[d, "\eta"'] & \mathcal{C}(c,c) \arrow[d, "j_c"] \\ \mathcal{D}(Fc, Fc) \arrow[r, "\varphi_{c,Fc}"'] & \displaystyle\sum_{x \in X} \mathcal{C}(c, x) \end{tikzcd}$$

The unit law states that applying η_{Fc} then the lifting operation is equal to applying η_c then the coproduct injection j_c.

ENRICHED COFUNCTORS

$$
\begin{tikzcd}
\mathcal{D}(Fc,d)\otimes\mathcal{D}(d,d') \arrow[r, "\varphi_{c,d}\otimes\mathrm{id}"] \arrow[dd, "\mu"'] & \left(\sum\limits_{x\in X}\mathcal{C}(c,x)\right)\otimes\mathcal{D}(d,d') \arrow[d, "\cong"] \\
 & \sum\limits_{x\in X}\mathcal{C}(c,x)\otimes\mathcal{D}(Fx,d') \arrow[d, "\sum(\mathrm{id}\otimes\varphi_{x,d'})"] \\
\mathcal{D}(Fc,d') \arrow[dd, "\varphi_{c,d'}"'] & \sum\limits_{x\in X}\mathcal{C}(c,x)\otimes\left(\sum\limits_{y\in Y}\mathcal{C}(x,y)\right) \arrow[dd, "\cong"] \\
 & \\
\sum\limits_{y\in Y}\mathcal{C}(c,y) & \sum\limits_{y\in Y}\left(\sum\limits_{x\in X}\mathcal{C}(c,x)\otimes\mathcal{C}(x,y)\right) \arrow[l, "\sum(\sum\mu)"]
\end{tikzcd}
$$

multiplication law

$X = F^{-1}(d)$

$Y = F^{-1}(d')$

COMPOSING ENRICHED COFUNCTORS

Given enriched cofunctors $(F,\varphi)\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $(G,\gamma)\colon \mathcal{D} \longrightarrow \mathcal{E}$, their composite enriched cofunctor is given by:

- The function $GF\colon \mathcal{C}_0 \longrightarrow \mathcal{E}_0$.
- The lifting operation:

$$\begin{tikzcd}
\mathcal{E}(GFc, e) \arrow[r, "\gamma_{Fc,e}"] \arrow[d] & \sum\limits_{x \in G^{-1}(e)} \mathcal{D}(Fc, x) \arrow[d, "\varphi_{c,x}"] \\
\sum\limits_{z \in (GF)^{-1}(e)} \mathcal{C}(c, z) & \sum\limits_{x \in G^{-1}(e)} \Big(\sum\limits_{y \in F^{-1}(x)} \mathcal{C}(c, y) \Big) \arrow[l, "\cong"']
\end{tikzcd}$$

DOUBLE CATEGORY OF ENRICHED COFUNCTORS

There is a double category $\mathcal{V}\text{-}\mathbb{C}\mathrm{of} := \mathrm{Mnd}_{\mathrm{ret}}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at})$ whose:

- objects are $\mathcal{V}$-categories;
- horizontal morphisms are $\mathcal{V}$-functors;
- vertical morphisms are $\mathcal{V}$-cofunctors
- cells with boundary,

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[d, "{(F,\varphi)}"'] & C \arrow[d, "{(G,\gamma)}"] \\ B \arrow[r, "K"'] & D \end{tikzcd}$$

satisfy the axioms:

$$\begin{tikzcd} A_0 \arrow[r, "H"] \arrow[d, "F"'] & C_0 \arrow[d, "G"] \\ B_0 \arrow[r, "K"'] & D_0 \end{tikzcd} \quad \text{in } \mathbf{Set}$$

$$\begin{tikzcd} B(Fa, b) \arrow[r, "\varphi_{a,b}"] \arrow[d, "K_{Fa,b}"'] & \sum_{x \in X} A(a, x) \arrow[d, "H_{a,x}"] \\ D(GHa, Kb) \arrow[r, "\gamma_{GHa,Kb}"'] & \sum_{y \in Y} C(Ha, y) \end{tikzcd}$$

in $\mathcal{V}$, where $X = F^{-1}(b)$ and $Y = G^{-1}(Kb)$.

ENRICHED LENSES

An **enriched lens** $(F,\varphi)\colon \mathcal{A} \to \mathcal{B}$ is a cell in $\mathcal{V}$-$\mathbb{C}\mathrm{of}$ of the form:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "{(F,\varphi)}"', "\shortmid" marking] & \mathcal{B} \arrow[d, equal] \\
\mathcal{B} \arrow[r, equal] & \mathcal{B}
\end{tikzcd}$$

That is, a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ and a $\mathcal{V}$-cofunctor $(F,\varphi)\colon \mathcal{A} \to \mathcal{B}$ such that:

$$\begin{tikzcd}
\mathcal{B}(Fa,b) \arrow[r, "\varphi_{a,b}"] \arrow[rr, bend right, "\mathrm{id}"'] & \sum\limits_{x \in X} \mathcal{A}(a,x) \arrow[r, "F_{a,x}"] & \mathcal{B}(Fa,b)
\end{tikzcd}$$

We obtain a double category **$\mathcal{V}$-$\mathbb{L}\mathrm{ens}$**.

ENRICHED CATEGORY OF LIFTS

Let $\mathcal{V}$ be an extensive category. Given a $\mathcal{V}$-cofunctor $(F,\varphi): \mathcal{A} \longrightarrow \mathcal{B}$, there is a $\mathcal{V}$-category Λ of lifts whose:

- set of objects $\Lambda_0 = \mathcal{A}_0$.
- hom-objects are given by:

$$\begin{tikzcd} \Lambda(a,a') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}(a,a') \arrow[d, "j_{a'}"] \\ \mathcal{B}(Fa,Fa') \arrow[r, "\varphi_{a,Fa'}"'] & \sum_{x \in X} \mathcal{A}(a,x) \end{tikzcd}$$

Call $F: \mathcal{A} \longrightarrow \mathcal{B}$ an enriched discrete opfibration if the morphism

$$\sum_{x \in X} \mathcal{A}(a,x) \longrightarrow \mathcal{B}(Fa,b)$$

is invertible for all $a \in \mathcal{A}_0$, $b \in \mathcal{B}_0$.

Thm: A $\mathcal{V}$-cofunctor $(F,\varphi): \mathcal{A} \longrightarrow \mathcal{B}$ is equivalent to a span in $\mathcal{V}$-$\mathcal{C}at$:

$$\begin{tikzcd} & \Lambda \arrow[dl, "\text{identity-on-objects}"'] \arrow[dr, "\text{enriched discrete opfibration}"] & \\ \mathcal{A} & & \mathcal{B} \end{tikzcd}$$

ENRICHMENT IN WEIGHTED SETS

A **weighted category** is a category enriched in wSet. In detail:

- A category $\mathcal{C}$ where each arrow $u: x \to y$ has a **weight** $|u| \in [0,\infty]$ such that:
  - *identity morphisms have weight zero : $|1_x| = 0$.
  - *for all composable morphisms $u: x \to y$ and $v: y \to z$ we have the **triangle inequality**: $|v \circ u| \leq |u| + |v|$

Example: A **Lawvere metric space** is a weighted category whose underlying category is codiscrete. Equivalently, it is a category enriched in $([0,\infty], +, 0)$.

WEIGHTED COFUNCTORS & LENSES

- A **weighted functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor (between the underlying categories) such that $|Fu| \leqslant |u|$ for all morphisms $u : c \to c'$ in $\mathcal{C}$.

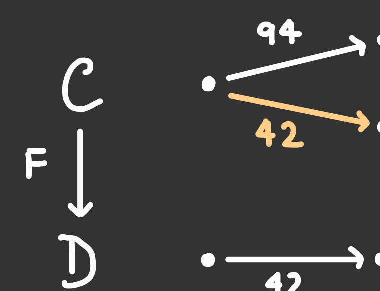
- A **weighted cofunctor** $(F, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ is a cofunctor with lifting operation $(c \in \mathcal{C}, u : Fc \to d \in \mathcal{D}) \longmapsto \varphi(c, u)$ such that $|\varphi(c,u)| \leqslant |u|$.

- A **weighted lens** $(F, \varphi) : \mathcal{C} \longrightarrow \mathcal{D}$ is a weighted functor and weighted cofunctor such that $F\varphi(a,u) = u$. We have that

$$|u| = |F\varphi(a,u)| \leqslant |\varphi(a,u)| \leqslant |u|$$

implies

$$|\varphi(a,u)| = |u|$$



FUTURE WORK & QUESTIONS

- Consider other bases of enrichment ($\mathcal{V} = \mathcal{C}at, \mathcal{L}ens, \mathcal{V}ect$) and possibility of arbitrary $\mathcal{V}$ (or bicategories/double categories)?
- Enriched split opfibrations? Using décalage?
- How much of the theory of lenses transfers to the enriched context?
- For which double categories can monad retro maps be represented as spans of monad maps?