Three approaches to lenses over a base

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MOTIVATION & OVERVIEW

$$
\begin{tikzcd}
 & \text{functor} \arrow[dl, "\text{property}"'] \arrow[dr, "\text{structure}"] & \\
\text{opfibration} \arrow[dr, "\text{structure}"'] & & \text{lens} \arrow[dl, "\text{property}"] \\
 & \text{split opfibration} &
\end{tikzcd}
$$

1. Fibred approach

$$\boxed{\mathcal{L}ens(B) \simeq [\mathbb{H}B, s\mathbb{M}\mathrm{ult}]_{\mathrm{lax}}}$$

2. Algebraic approach

$$\boxed{\mathcal{L}ens(B) \xrightarrow{\text{monadic}} \mathcal{C}at/B}$$

3. Coalgebraic approach

$$\boxed{\mathcal{L}ens(B) \xrightarrow{\text{comonadic}} \mathcal{C}of(B)}$$

BACKGROUND: WHAT IS A LENS?

A **lens** is a functor equipped with a suitable choice of lifts.

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\varphi(a,u)"] \arrow[d, dash, dotted] & a' \arrow[d, dash, dotted] \\ fa \arrow[r, "u"] & b \end{tikzcd}$$

A **split opfibration** is a lens whose chosen lifts are **opcartesian**.

Proposition: A lens $A \xrightarrow{(f,\varphi)} B$ $\simeq$ a commutative diagram of functors,

$$\begin{tikzcd} A \arrow[dd, "f"'] & \\ & X \arrow[ul, "\varphi"'] \arrow[dl, "\overline{\varphi}"] \\ B & \end{tikzcd}$$

where φ is **bijective-on-objects** and $\overline{\varphi}$ is a **discrete opfibration**.

THE CATEGORY OF LENSES OVER A BASE

For each small category B, there is a category $\mathcal{L}ens(B)$ whose:

- objects are lenses into B;
- morphisms are given by:

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[ddr, bend right, "f"] & & C \arrow[ddl, bend left, "g"'] \\
X \arrow[u, "\varphi"] \arrow[rr] \arrow[dr, "\bar{\varphi}"'] & & Y \arrow[u, "\gamma"'] \arrow[dl, "\bar{\gamma}"] \\
& B &
\end{tikzcd}$$

i.e. functors $h : A \to C$ such that $gh = f$ & $h\varphi(a,u) = \gamma(ha,u)$.

- This defines a functor:

$$\mathcal{L}ens(-) : \mathcal{C}at^{op} \longrightarrow \mathrm{CAT}$$

- There are forgetful functors:

$$\mathcal{L}ens(B) \longrightarrow \mathcal{C}at/B$$

$$\mathcal{L}ens(B) \longrightarrow \mathcal{C}of(B)$$

- There are full subcategories:

$$\begin{tikzcd}
\mathrm{DOpf}(B) \arrow[r, hook] \arrow[r, phantom, shift left=3, "\top"] & \mathcal{L}ens(B) \arrow[l, dashed, bend right, shift right=2]
\end{tikzcd}$$

$$\mathrm{SOpf}(B) \hookrightarrow \mathcal{L}ens(B)$$

FIBRED APPROACH TO LENSES

Many kinds of morphism can be understood "fibre-wise":

$$\mathsf{SOpf}(B) \simeq [B, \mathcal{C}at]$$

$$\mathsf{DOpf}(B) \simeq [B, \mathcal{S}et]$$

$$\mathcal{C}at/B \simeq [\mathbb{H}B, \mathbb{S}\mathrm{pan}]_{lax}$$

Can we find a double category $\mathbb{D}$ which classifies lenses?

$$\mathcal{L}ens(B) \simeq [\mathbb{H}B, \mathbb{D}]_{lax}$$

Let $\mathbb{s}\mathbb{M}\mathrm{ult}$ be the double category of sets, functions & split multi-valued functions.

$$\begin{tikzcd} A \arrow[r, shift left, tail] & X \arrow[l, shift left, two heads] \arrow[r] & B \end{tikzcd}$$

There are identity-on-objects & vert. arrows double functors:

$$\begin{tikzcd} \mathbb{Q}\mathrm{Set} \arrow[r, shift left=2, hook, "L"] \arrow[r, phantom, "\perp"] & \mathbb{s}\mathbb{M}\mathrm{ult} \arrow[l, shift left=2, "R"] \arrow[r] & \mathbb{S}\mathrm{pan} \end{tikzcd}$$

FIBRED APPROACH TO LENSES

Theorem: There is an equivalence:

$$\mathcal{L}ens(B) \simeq [\mathbb{H}B, s/Mult]_{lax}$$

Idea: Lax functor $\rightsquigarrow$ lens

$$\begin{tikzcd}
\int LRF \arrow[r, "\text{b.o.o.}"] \arrow[dr] & \int F \arrow[r] \arrow[d, "\text{lens}"'] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "*"] \\
& B \arrow[r, "LRF"] \arrow[r, bend right=60, "F"'] \arrow[r, phantom, shift right=4, "\Downarrow \varepsilon F"] & s/Mult
\end{tikzcd}$$

discrete opfibration

Split opfibration $\simeq$ lax functor $F : \mathbb{H}B \longrightarrow s/Mult$ such that

$$\begin{tikzcd}
F_x \arrow[r, "LRF(u)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \varepsilon_{F(u)}"] & F_y \arrow[r, "F(1_y)"] \arrow[d, equal] & F_y \arrow[d, equal] \\
F_x \arrow[r, "F(u)"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \mathrm{comp}(u, 1_y)"] & F_y \arrow[r, "F(1_y)"] & F_y \arrow[d, equal] \\
F_x \arrow[rr, "F(u)"'] & & F_y
\end{tikzcd}$$

is an isocell for all $u : x \to y \in B$.

6

ALGEBRAIC APPROACH TO LENSES

Examples of lenses are morphisms with algebraic structure:

$$\mathrm{SOpf}(B) \xrightarrow{\text{monadic}} \mathrm{Cat}/B$$

$$\mathrm{DOpf}(B) \xrightarrow{\text{monadic}} \mathrm{Set}/B_0$$

Leads to generalisations:

- Cat $\rightsquigarrow$ 2-category with pullbacks and comma objects.
- Set $\rightsquigarrow$ category with pullbacks.

Johnson-Rosebrugh (2013): Lenses are certain algebras for a semi-monad:

$$\mathrm{Cat}/B \longrightarrow \mathrm{Cat}/B$$

$$A \xrightarrow{f} B \longmapsto f i_A \downarrow B \xrightarrow{\pi} B$$

Can we show that

$$\mathrm{Lens}(B) \longrightarrow \mathrm{Cat}/B$$

is monadic? Can we generalise lenses by replacing Cat?

ALGEBRAIC APPROACH TO LENSES

Recall that $\mathcal{C}at$ has:

- An idempotent comonad (disc. objs.).
- An O.F.S. (initial, disc. opfibration).

Theorem: $\mathcal{L}ens(B) \xrightarrow{\text{monadic}} \mathcal{C}at/B$

Idea:

$$
\begin{tikzcd}
A_0 \arrow[r, two heads] \arrow[d, "s"', "\text{b.o.o.}"'] & I \arrow[d, "s"] \arrow[ddr, bend left, "\text{discrete opfibration}"] & \\
A \arrow[r] \arrow[drr, bend right, "f"'] & \bullet \arrow[dr, dashed, "\text{lens}"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & B
\end{tikzcd}
$$

Corollary:

$$
\begin{tikzcd}
\mathcal{S}\mathcal{O}pf(B) \arrow[rr, hook] \arrow[dr, shift left] & & \mathcal{L}ens(B) \arrow[ll, bend right, dashed] \arrow[dl, shift left] \arrow[ll, phantom, "\perp", bend right=15] \\
& \mathcal{C}at/B \arrow[ul, shift left] \arrow[ur, shift left] \arrow[ul, phantom, "\top"] \arrow[ur, phantom, "\top"] &
\end{tikzcd}
$$

Bonus: Can replace $\mathcal{C}at$ with $\mathcal{C}$, category t/w idem. comonad & O.F.S.

Example: $\mathcal{S}p\mathcal{E}pi(B) \xrightarrow{\text{monadic}} \mathcal{C}/B$

- The initial object comonad.
- The O.F.S. (all morphisms, iso).

COALGEBRAIC APPROACH TO LENSES

There is a category Cof(B) whose:

- objects are cofunctors into B;
- morphisms are commutative diagrams:

$$\begin{tikzcd}
A \arrow[rr, "h"] & & C \\
X \arrow[u, "\varphi"] \arrow[rr] \arrow[dr, "\bar{\varphi}"'] & & Y \arrow[u, "\gamma"'] \arrow[dl, "\bar{\gamma}"] \\
& B &
\end{tikzcd}$$

b.o.o. b.o.o.

disc. opfibration disc. opfibration

Can we show that

$$\mathcal{L}ens(B) \longrightarrow Cof(B)$$

is comonadic? Generalisation?

Lemma: $\mathcal{L}ens(B) \simeq Cof(B)/1_B$

Idea:

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[d, equal] \\
X \arrow[u, "\varphi"] \arrow[r, "\bar{\varphi}"] \arrow[d, "\bar{\varphi}"'] & B \arrow[dl, equal] \\
B &
\end{tikzcd}$$

COALGEBRAIC APPROACH TO LENSES

Recall that $\mathcal{C}at$ has:

- An idempotent monad (codisc. objs.).

Theorem: $\mathcal{L}ens(B) \xrightarrow{\text{comonadic}} \mathcal{C}of(B)$

Idea:

$$
\begin{tikzcd}
X \arrow[dr, dashed, "\wr"'] \arrow[drr, bend left, "\bar{\varphi}"] \arrow[ddr, bend right, "\varphi"'] & & \\
& \bullet \arrow[r] \arrow[d, "\wr"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\wr"] \\
& A \arrow[r] & B_\infty
\end{tikzcd}
$$

discrete opfibration

lens

b.o.o.

b.o.o.

codisc. cat.

Question: Is the functor

$\mathcal{S}\mathcal{O}pf(B) \longrightarrow \mathcal{C}of(B)$ comonadic?

Bonus: Can replace $\mathcal{C}at$ with $\mathcal{C}$, category t/w idempotent monad and class $\mathcal{M}$ of morphisms.

Example: $\mathcal{S}p\mathcal{E}pi(B) \xrightarrow{\text{comonadic}} B/\mathcal{C}$

- The terminal object monad.
- $\mathcal{M}$ = class of identity morphisms

SUMMARY & FUTURE WORK

1. Fibred approach

$$\boxed{\mathcal{L}\mathrm{ens}(B) \simeq [\mathbb{H}B, \mathsf{s}\mathbb{M}\mathrm{ult}]_{\mathrm{lax}}}$$

2. Algebraic approach

$$\boxed{\mathcal{L}\mathrm{ens}(B) \xrightarrow{\text{monadic}} \mathcal{C}\mathrm{at}/B}$$

3. Coalgebraic approach

$$\boxed{\mathcal{L}\mathrm{ens}(B) \xrightarrow{\text{comonadic}} \mathcal{C}\mathrm{of}(B)}$$

- Understand double cat. $\mathsf{s}\mathbb{M}\mathrm{ult}$
- Fibred approach to $\mathcal{C}\mathrm{of}(B)$?
- Duality between (co)algebras:

$$\begin{tikzcd} \mathcal{C}\mathrm{at}/B \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathcal{L}\mathrm{ens}(B) \arrow[l, shift right] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathcal{C}\mathrm{of}(B) \arrow[l, shift right] \end{tikzcd}$$

- When do morphisms with algebraic structure compose?
- A formal theory of lenses

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