

THREE APPROACHES TO LENSES OVER A BASE

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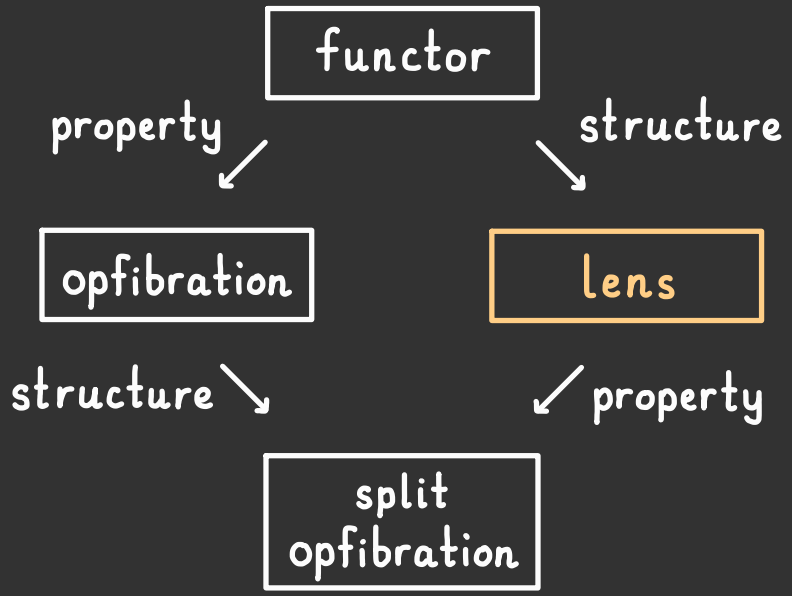
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MOTIVATION & OVERVIEW



1. Fibred approach

$$\mathcal{L}ens(B) \simeq [HB, s/Mult]_{lax}$$

2. Algebraic approach

$$\mathcal{L}ens(B) \xrightarrow{\text{monadic}} \mathcal{C}at/B$$

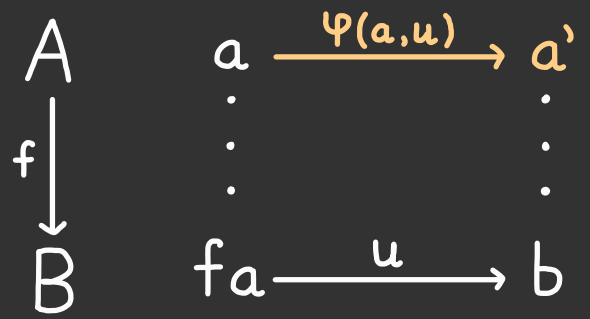
3. Coalgebraic approach

$$\mathcal{L}ens(B) \xrightarrow{\text{comonadic}} \mathcal{C}of(B)$$

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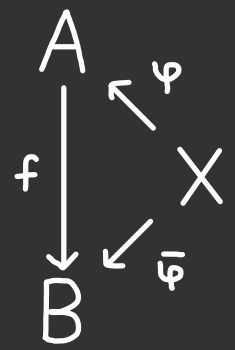
BACKGROUND: WHAT IS A LENS?

A **lens** is a functor equipped with a suitable choice of lifts.



A **split opfibration** is a lens whose chosen lifts are **opcartesian**.

Proposition: A lens $A \xrightarrow{(f,\varphi)} B \simeq$ a commutative diagram of functors,



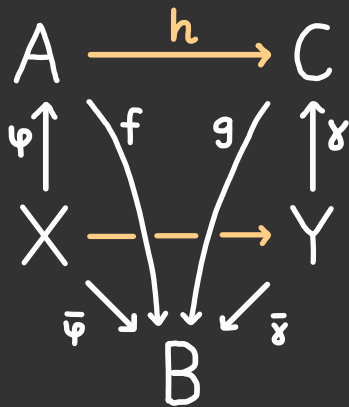
where φ is **bijective-on-objects** and $\bar{\varphi}$ is a **discrete opfibration**.

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THE CATEGORY OF LENSES OVER A BASE

For each small category B , there is a category $\mathcal{Lens}(B)$ whose:

- objects are lenses into B ;
- morphisms are given by:



i.e. functors

$h: A \rightarrow C$ such

that $gh = f$ &

$h\psi(a, u) = \chi(ha, u)$.

- This defines a functor:

$$\mathcal{Lens}(-) : \mathcal{Cat}^{\text{op}} \longrightarrow \mathcal{CAT}$$

- There are forgetful functors:

$$\mathcal{Lens}(B) \longrightarrow \mathcal{Cat}/B$$

$$\mathcal{Lens}(B) \longrightarrow \mathcal{Cof}(B)$$

- There are full subcategories:

$$\mathcal{DOpf}(B) \xrightarrow{\tau} \mathcal{Lens}(B)$$

$$\mathcal{SOpf}(B) \hookrightarrow \mathcal{Lens}(B)$$

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FIBRED APPROACH TO LENSES

Many kinds of morphism can be understood "fibre-wise":

$$\text{SOpf}(B) \cong [B, \text{Cat}]$$

$$\text{DOpf}(B) \cong [B, \text{Set}]$$

$$\text{Cat}/B \cong [\text{IB}, \text{Span}]_{\text{Iax}}$$

Can we find a double category ID which classifies lenses?

$$\text{Lens}(B) \cong [\text{IB}, \text{ID}]_{\text{Iax}}$$

Let s/Mult be the double category of sets, functions & split multi-valued functions.

$$A \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} X \longrightarrow B$$

There are identity-on-objects & vert. arrows double functors:

$$\text{QSet} \begin{array}{c} \xleftarrow{L} \\ \perp \\ \xrightarrow{R} \end{array} s/\text{Mult} \longrightarrow \text{Span}$$

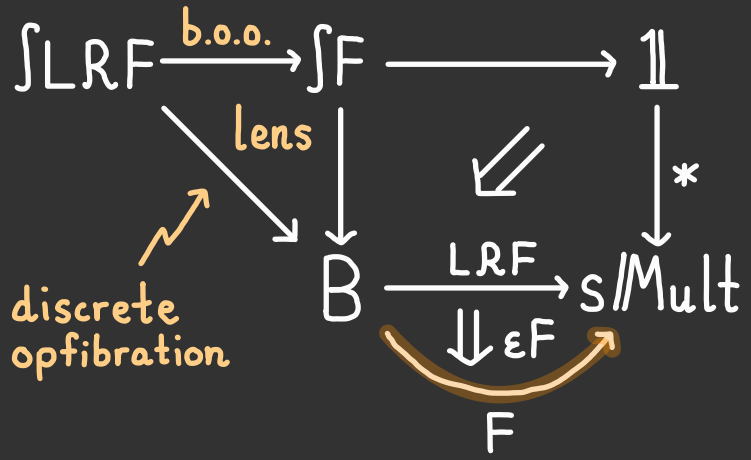
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FIBRED APPROACH TO LENSES

Theorem: There is an equivalence:

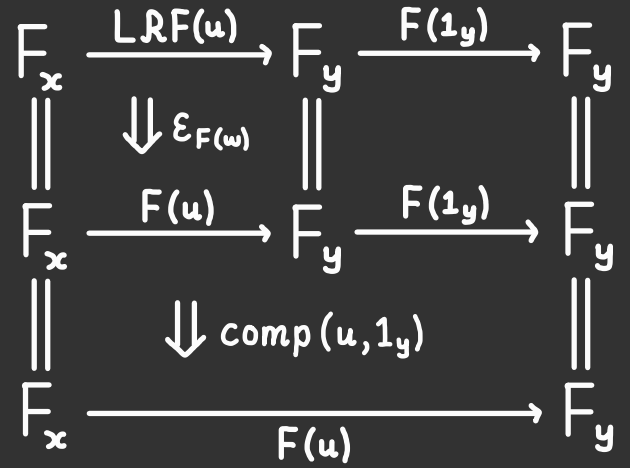
$$\text{Lens}(B) \cong [\text{HB}, \text{s/Mult}]_{\text{lax}}$$

Idea: Lax functor \rightsquigarrow Lens



Split opfibration \cong lax functor

$F: \text{HB} \rightarrow \text{s/Mult}$ such that



is an isocell for all $u: x \rightarrow y \in B$.

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ALGEBRAIC APPROACH TO LENSES

Examples of lenses are morphisms with algebraic structure:

$$S\text{Opf}(B) \xrightarrow{\text{monadic}} \text{Cat}/B$$

$$D\text{Opf}(B) \xrightarrow{\text{monadic}} \text{Set}/B_0$$

Leads to generalisations:

- $\text{Cat} \rightsquigarrow$ 2-category with pullbacks and comma objects.
- $\text{Set} \rightsquigarrow$ category with pullbacks.

Johnson-Rosebrugh (2013): Lenses are certain algebras for a semi-monad:

$$\text{Cat}/B \longrightarrow \text{Cat}/B$$

$$A \xrightarrow{f} B \longmapsto f_{i_A} \downarrow B \xrightarrow{\pi} B$$

Can we show that

$$\text{Lens}(B) \longrightarrow \text{Cat}/B$$

is **monadic**? Can we generalise lenses by replacing Cat ?

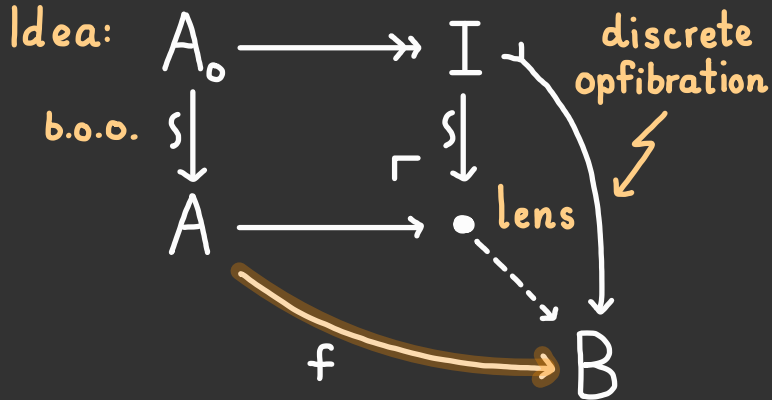
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ALGEBRAIC APPROACH TO LENSES

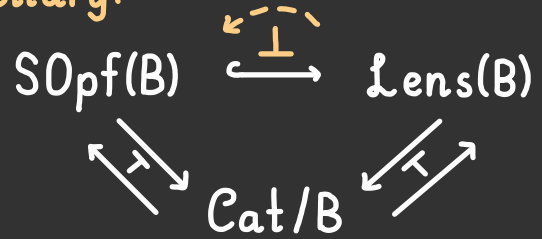
Recall that Cat has:

- An idempotent comonad (disc. objs.).
- An O.F.S. (initial, disc. opfibration).

Theorem: $\text{Lens}(B) \xrightarrow{\text{monadic}} \text{Cat}/B$



Corollary:



Bonus: Can replace Cat with \mathcal{C} , category t/w idem. comonad & O.F.S.

Example: $\text{SpEpi}(B) \xrightarrow{\text{monadic}} \mathcal{C}/B$

- The initial object comonad.
- The O.F.S. (all morphisms, iso).

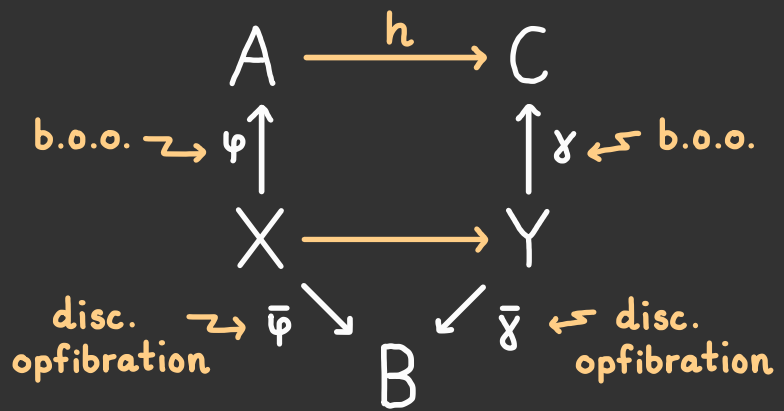
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COALGEBRAIC APPROACH TO LENSES

There is a category $\text{Cof}(B)$ whose:

- objects are **cofunctors** into B ;
- morphisms are commutative

diagrams:



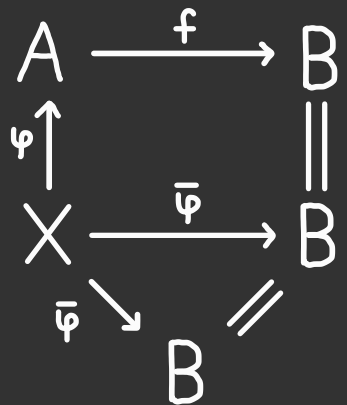
Can we show that

$$\text{Lens}(B) \longrightarrow \text{Cof}(B)$$

is **comonadic**? Generalisation?

Lemma: $\text{Lens}(B) \simeq \text{Cof}(B)/1_B$

Idea:



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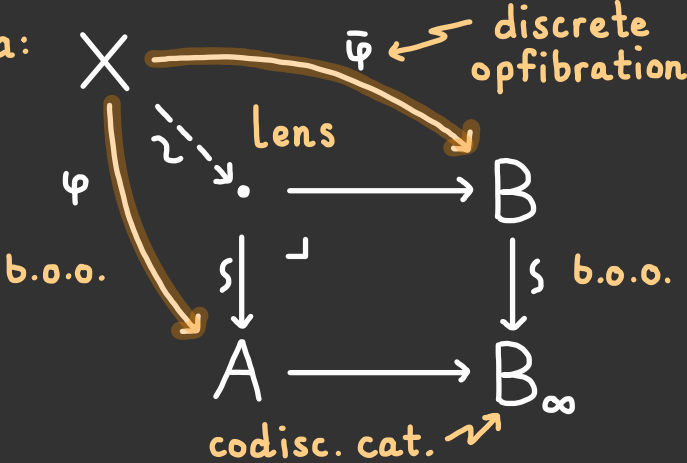
COALGEBRAIC APPROACH TO LENSES

Recall that Cat has:

- An idempotent monad (codisc. obj.).

Theorem: $\text{Lens}(B) \xrightarrow{\text{comonadic}} \text{Cof}(B)$

Idea:



Question: Is the functor $\text{SOpf}(B) \rightarrow \text{Cof}(B)$ comonadic?

Bonus: Can replace Cat with \mathcal{C} , category t/w idempotent monad and class \mathcal{M} of morphisms.

Example: $\text{SpEpi}(B) \xrightarrow{\text{comonadic}} B/\mathcal{C}$

- The terminal object monad.
- $\mathcal{M} =$ class of identity morphisms

SUMMARY & FUTURE WORK

1. Fibred approach

$$\mathcal{L}ens(B) \simeq [IH\mathcal{B}, s/Mult]_{lax}$$

2. Algebraic approach

$$\mathcal{L}ens(B) \xrightarrow{\text{monadic}} \mathcal{C}at/B$$

3. Coalgebraic approach

$$\mathcal{L}ens(B) \xrightarrow{\text{comonadic}} \mathcal{C}of(B)$$

- Understand double cat. $s/Mult$
- Fibred approach to $\mathcal{C}of(B)$?
- Duality between (co)algebras:
 $\mathcal{C}at/B \xrightleftharpoons{\perp} \mathcal{L}ens(B) \xrightleftharpoons{\perp} \mathcal{C}of(B)$
- When do morphisms with algebraic structure compose?
- A formal theory of lenses

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