

THE DOUBLE CATEGORY OF  
GENERALISED LENSES

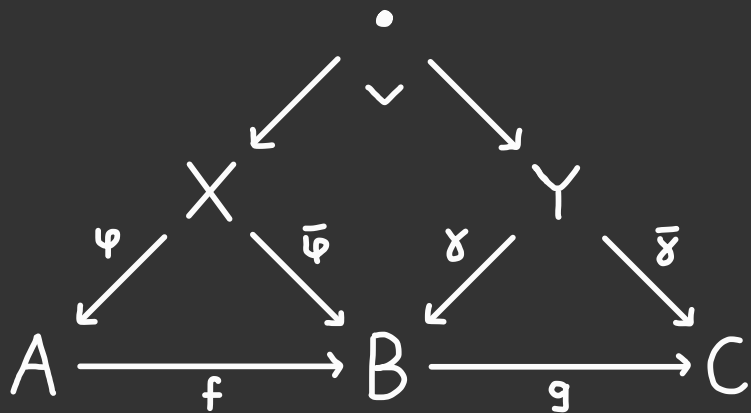
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AUSTRALIAN CATEGORY SEMINAR  
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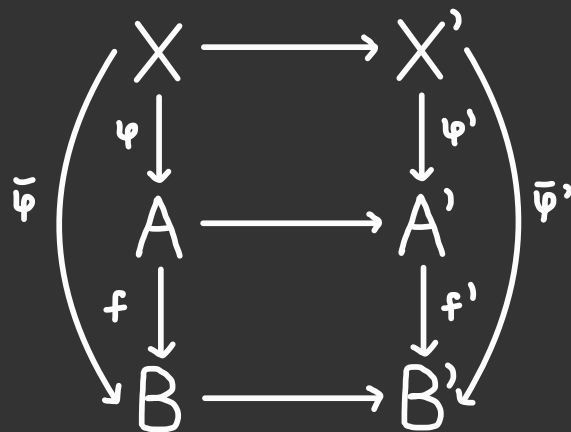
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# MOTIVATION

Lenses as morphisms



Lenses as objects

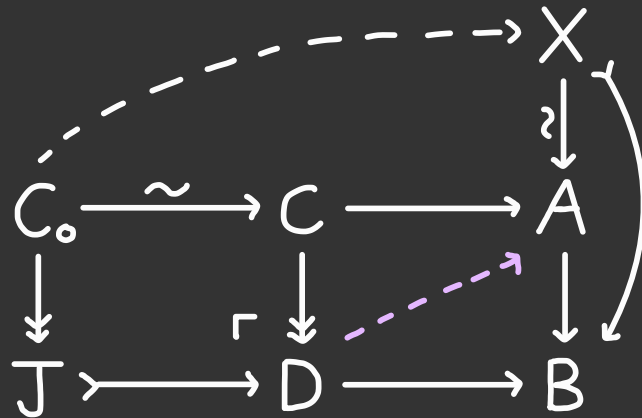


- How can we bridge these two approaches to lenses?
- What is the general setting in which lenses should be considered?
- Why is lifting an intrinsic aspect of lenses?

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# OVERVIEW OF THE TALK

1. Background and generalised lenses
2. Algebraic weak factorisation systems



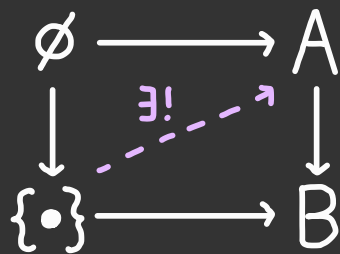
3. The double category of generalised lenses

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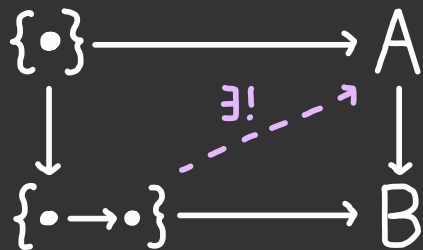
## BACKGROUND

A functor  $f: A \rightarrow B$  is called:

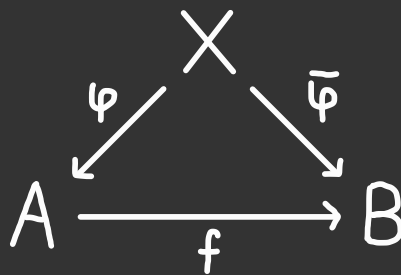
- **bijective-on-objects** if we have:



- a **discrete opfibration** if we have:



A **delta lens**  $(f, \varphi): A \rightarrow B$  is a commutative diagram in  $\mathcal{Cat}$ ,



where  $\varphi$  is **bijective-on-objects** and  $\bar{\varphi}$  is a **discrete opfibration**.

**Question:** How can we interpret this diagram in other categories?

## 4 IDEMPOTENT COMONADS & DISCRETE OBJECTS

- Let  $(-)_o: \mathcal{C} \rightarrow \mathcal{C}$  be an idempotent comonad with counit  $i_A: A_o \rightarrow A$ .
- An object  $X$  is **discrete** if  $X \cong A_o$ .
- The full subcategory  $\text{Disc}(\mathcal{C})$  on discrete objects is coreflective.
- A morphism  $f: A \rightarrow B$  is a **weak equivalence** if  $f_o$  is an iso; will be denoted:  $\bullet \xrightarrow{\sim} \bullet$

- The class  $\text{Iso}(\mathcal{C}) \subseteq \mathcal{W} \subseteq \mathcal{C}$  of weak equivalences satisfies 2-out-of-3 and is stable under pullback (& pushouts\*).
- Universal property:

$$\begin{array}{ccc} & & X \\ & \exists! \nearrow & \downarrow \wr \\ A_o & \longrightarrow & B \end{array}$$

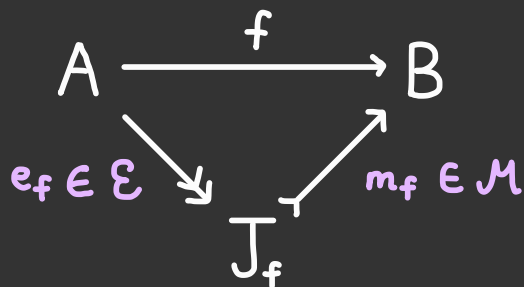
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# ORTHOGONAL FACTORISATION SYSTEMS

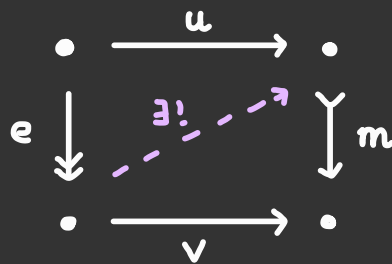
• Let  $\mathcal{C}$  be a category t/w classes of morphisms  $\text{Iso}(\mathcal{C}) \subseteq \mathcal{E}, \mathcal{M} \subseteq \mathcal{C}$  which are closed under composition.

• The pair  $(\mathcal{E}, \mathcal{M})$  is an **orthogonal factorisation system** on  $\mathcal{C}$  if:

– every morphism decomposes:



– the following universal property holds:



• Several nice properties:

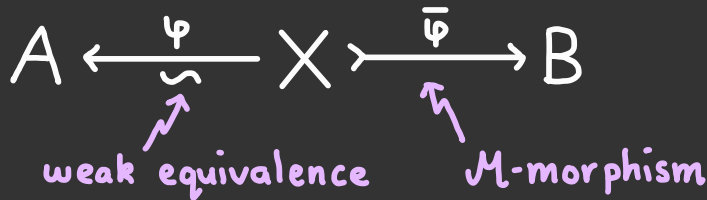
– if  $gf \in \mathcal{M}$  and  $g \in \mathcal{M}$ , then  $f \in \mathcal{M}$

– morphisms in  $\mathcal{M}$  are stable under pullback.

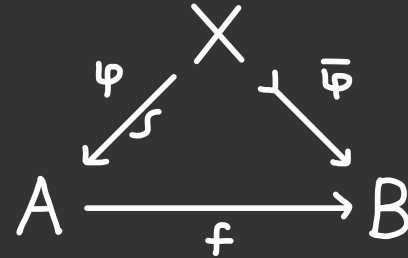
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# COMORPHISMS & GENERALISED LENSES

- Let  $\mathcal{C}$  be a category with,
  - an idempotent comonad  $(-)_\circ$  yielding class  $\mathcal{W}$  of weak equiv.
  - an O.F.S. given by  $(\mathcal{E}, \mathcal{M})$  such that  $\mathcal{W} \cap \mathcal{M} = \text{Iso}(\mathcal{C})$  and  $\text{Disc}(\mathcal{C}) \hookrightarrow \mathcal{M}$ .
- A comorphism  $\varphi: A \twoheadrightarrow B$  is an isomorphism class of spans:



- A generalised lens  $(f, \varphi): A \rightarrow B$  is a morphism  $f: A \rightarrow B$  and a comorphism  $\varphi: A \twoheadrightarrow B$  such that the following diagram commutes:



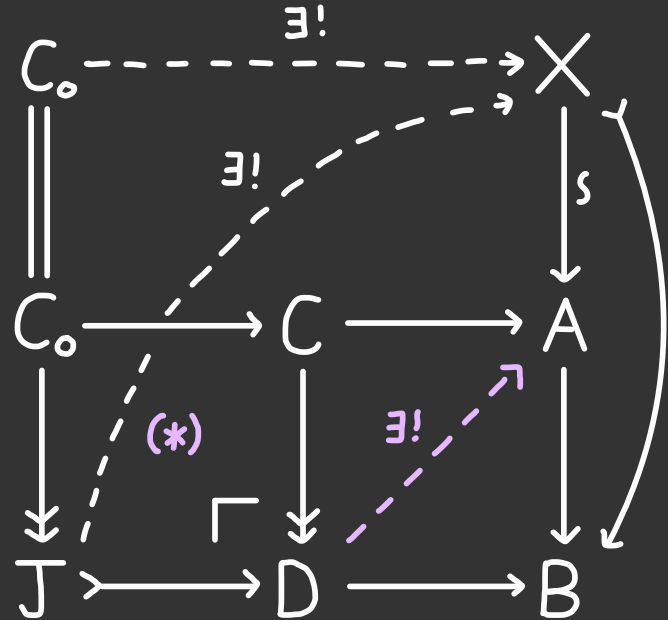
Question: Which lifting problems do generalised lenses solve?

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# LIFTING FOR GENERALISED LENSES

A step-by-step construction:

1. Consider a lifting problem whose right side is a lens
2. Precompose with counit and use universal property of w.e.
3. Factorise the composite and use universal property of O.F.S.
4. Suppose the square is a pushout and use universal property.
5. Rest.



**Takeaway:** Lenses have the right lifting property against morphisms satisfying (\*).



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# INITIAL FUNCTORS

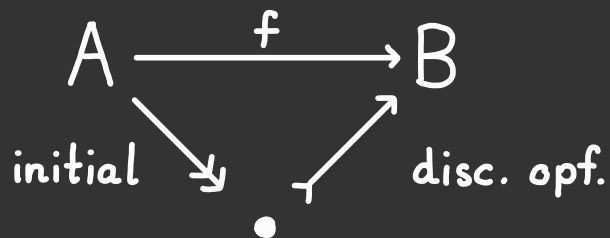
A functor  $f:A \rightarrow B$  is called *initial* if any of the following conditions hold:

- 1) for all  $b \in B$ ,  $f/b$  is connected.
- 2) for all  $g:B \rightarrow C$ ,  $\lim(g) \simeq \lim(g \circ f)$ .

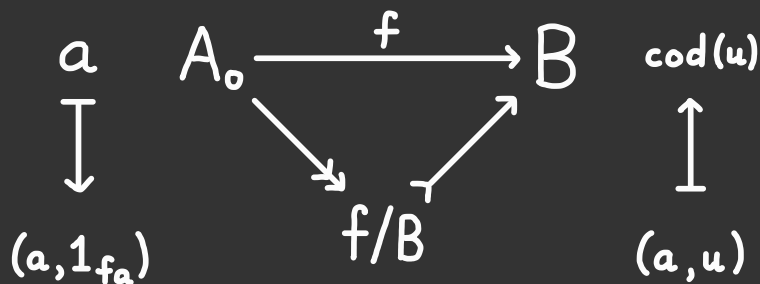
Examples:

- A functor  $1 \rightarrow B$  is initial  $\Leftrightarrow$  it chooses an *initial object* in  $B$ .
- A functor  $A_0 \rightarrow B$  is initial  $\Leftrightarrow$  it chooses *local initial objects* in  $B$ .
- Left adjoint functors.

The *comprehensive factorisation system* decomposes every functor:



Example: The comprehensive factorisation of  $f:A_0 \rightarrow B$  is:



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# UNPACKING LIFTS FOR DELTA LENSES

Consider a lifting problem:

$$\begin{array}{ccc}
 \{*\} & \xrightarrow{a} & A \\
 \downarrow & & \downarrow f \\
 \text{initial} & & \\
 \text{obj.} & & \\
 f_a/B & \xrightarrow{\text{cod}} & B
 \end{array}$$

A solution  $p: f_a/B \rightarrow A$  states:

- $p(a, 1_{f_a}) = a$  &  $f_p(a, u) = \text{cod}(u)$
- $p\langle 1_a, 1_{f_a} \rangle = 1_a$  &  $f_p\langle 1_a, u \rangle = u$
- $p\langle 1_a, v \circ u \rangle = p\langle 1_a, v \rangle \circ p\langle 1_a, u \rangle$

Then  $f$  has a local lens structure at the object  $a \in A$ .

- The coproduct over all  $a \in A$  is a solution, but doesn't give a (global) delta lens structure.
- Conversely, the functors which lift against delta lenses are relatively easy to understand.

$$\begin{array}{ccc}
 C_0 & \xrightarrow{i_c} & C \\
 \downarrow & & \downarrow \\
 f_{i_c}/D & \xrightarrow{\quad} & D
 \end{array}$$

Paste local initial objects along an object inclusion

# 10 ALGEBRAIC WEAK FACTORISATION SYSTEMS

An algebraic weak factorisation system (AFWS) on  $\mathcal{C}$  consists of:

- A functorial factorisation  $(L, E, R)$

$$\begin{array}{ccccc}
 & & f & & \\
 & \curvearrowright & & \curvearrowleft & \\
 X & \xrightarrow{L_f} & E_f & \xrightarrow{R_f} & Y \\
 h \downarrow & & \downarrow E(h,k) & & \downarrow k \\
 W & \xrightarrow{L_g} & E_g & \xrightarrow{R_g} & Z \\
 & \curvearrowleft & & \curvearrowright & \\
 & & g & & 
 \end{array}$$

- An extension of  $(L, \varepsilon)$  to a comonad  $(L, \varepsilon, \Delta)$ .

- An extension of  $(R, \eta)$  to a monad  $(R, \eta, \mu)$ .

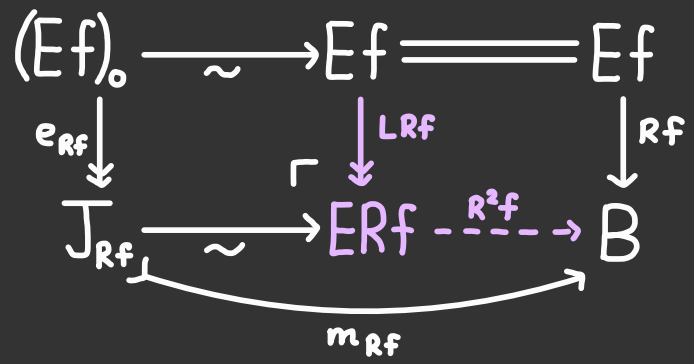
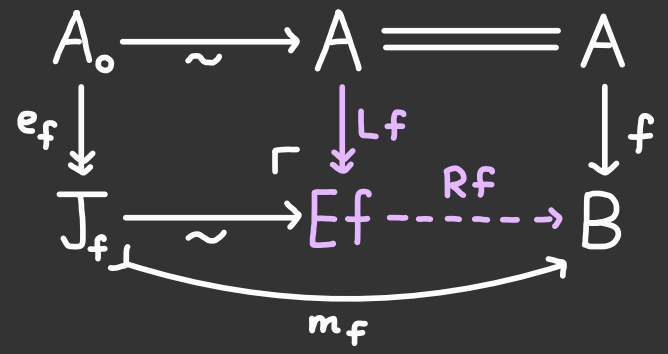
$$\begin{array}{ccccc}
 X & \xlongequal{\quad} & X & \xrightarrow{L_f} & E_f \\
 L_f \downarrow & \text{counit } \varepsilon_f & \downarrow f & \text{unit } \eta_f & \downarrow R_f \\
 E_f & \xrightarrow{R_f} & Y & \xlongequal{\quad} & Y
 \end{array}$$

- A distributive law  $\delta: LR \Rightarrow RL$  of the comonad  $L$  over the monad  $R$ :

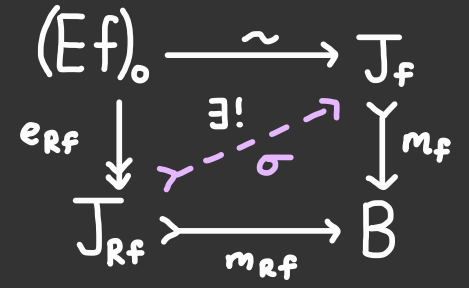
$$\begin{array}{ccc}
 E_f & \xrightarrow{\Delta_f} & EL_f \\
 LR_f \downarrow & \delta_f & \downarrow RL_f \\
 ER_f & \xrightarrow{M_f} & E_f
 \end{array}$$

# THE MONAD FOR GENERALISED LENSES...

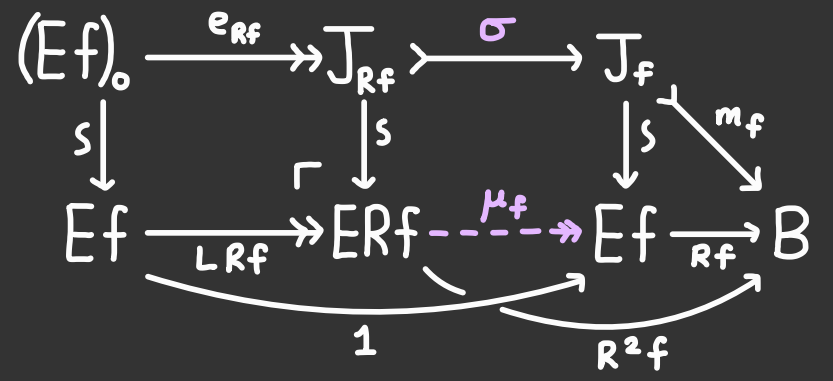
Let  $(\mathcal{C}, \mathcal{W}, \mathcal{E}, \mathcal{M})$  be as before,  
and assume  $\mathcal{C}$  has pushouts\*



Since we have,

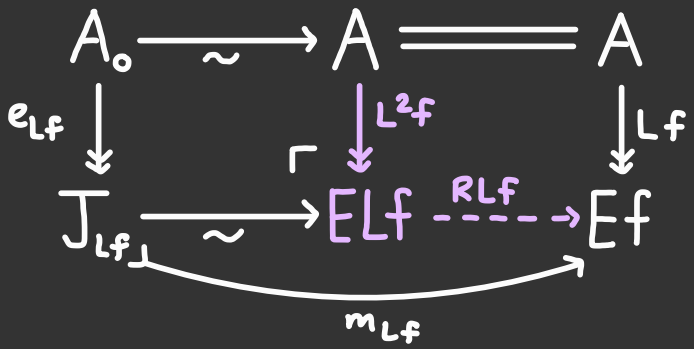


the multiplication is given by:

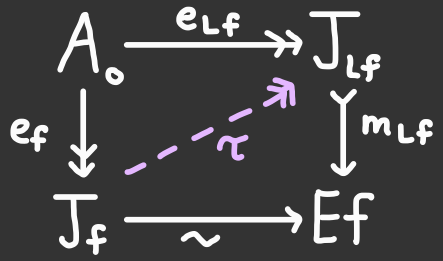


**12** ... AND THE CORRESPONDING A.W.F.S.

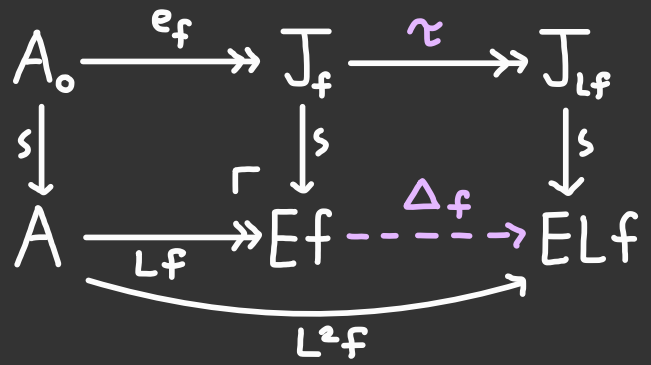
To construct the comonad,



note that we have:



The comultiplication is given by:



Altogether this gives the data for an AWFS on  $\mathcal{C}$ , constructed from the OFS  $(\mathcal{E}, \mathcal{M})$  and  $\mathcal{W}$ .

Question: What are the R-algebras?

# UNPACKING THE R-ALGEBRAS

Consider the monad  $(R, \eta, \mu)$  on  $\mathcal{C}^2$ .

- A  $R$ -algebra on  $f: A \rightarrow B$  is given by  $\hat{p}: Ef \rightarrow A$  such that  $f \circ \hat{p} = Rf$  and other axioms.

- $R$ -algebras are equivalent to:

$$\begin{array}{ccc}
 A_0 & \xrightarrow{\sim} & A \\
 e_f \downarrow & \nearrow p & \downarrow f \\
 J_f & \xrightarrow{m_f} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 J_{m_f} & \xrightarrow{P_2} & J_f \\
 \psi \downarrow & & \downarrow p \\
 J_f & \xrightarrow{p} & A
 \end{array}$$

where we have:

$$\begin{array}{ccccc}
 (J_f)_0 & \xrightarrow{\sim} & J_f & \xleftarrow{e_f} & A_0 & \xleftarrow{p_0} & (J_f)_0 \\
 e_{m_f} \downarrow & \nearrow \psi & \downarrow m_f & & & \nwarrow P_2 & \downarrow e_{m_f} \\
 J_{m_f} & \xrightarrow{m_f^2} & B & \xleftarrow{m_f^2} & J_{m_f} & & 
 \end{array}$$

For any AWFS on  $\mathcal{C}$ , there is a double category  $R\text{-Alg}$  whose:

- Category of objects is  $\mathcal{C}$ .
- Category of morphisms is the category of  $R$ -algebras.

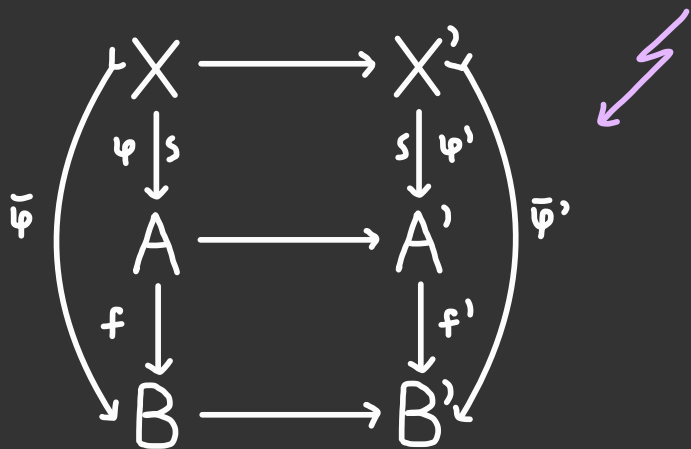
**Claim:** The double cat of lenses is isomorphic to  $R\text{-Alg}$ .

# 14 DOUBLE CATEGORY OF GENERALISED LENSES

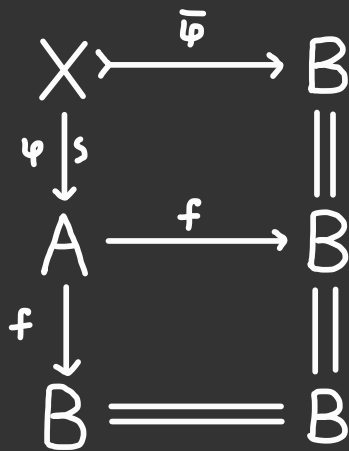
Let  $(\mathcal{C}, \mathcal{W}, \mathcal{E}, \mathcal{M})$  be as before,  
and assume  $\mathcal{C}$  has pullbacks\*

The double category  $\mathbb{L}ens(\mathcal{C})$  has:

- Category of objects given by  $\mathcal{C}$ .
- Category of morphisms  $\mathcal{V}Lens(\mathcal{C})$



The double category  $\mathbb{L}ens(\mathcal{C})$   
is right-connected since we  
have cells for each lens.

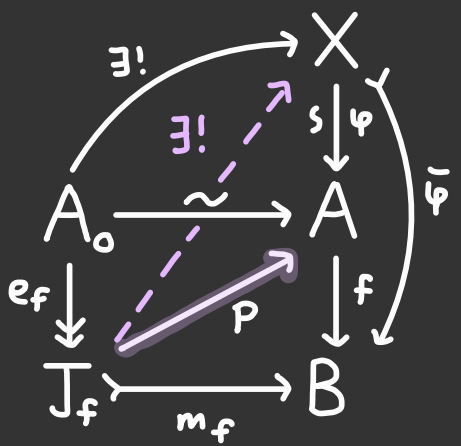


For  $\mathbb{L}ens(\mathcal{C}) \cong R\text{-Alg}$ , we need  
to show that  $\mathcal{V}Lens(\mathcal{C}) \xrightarrow{\text{monadic}} \mathcal{C}^2$

# MONADICITY

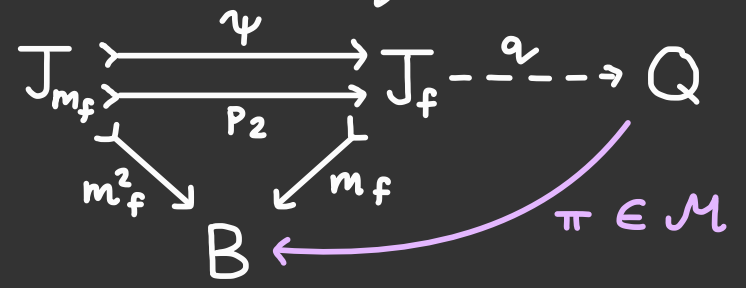
Let  $(R, \eta, \mu)$  be the monad on  $\mathcal{C}^2$  as before. We want  $vLens(\mathcal{C})$  to be isomorphic to  $R\text{-Alg}$ .

Step 1:  $vLens(\mathcal{C}) \longrightarrow R\text{-Alg}$

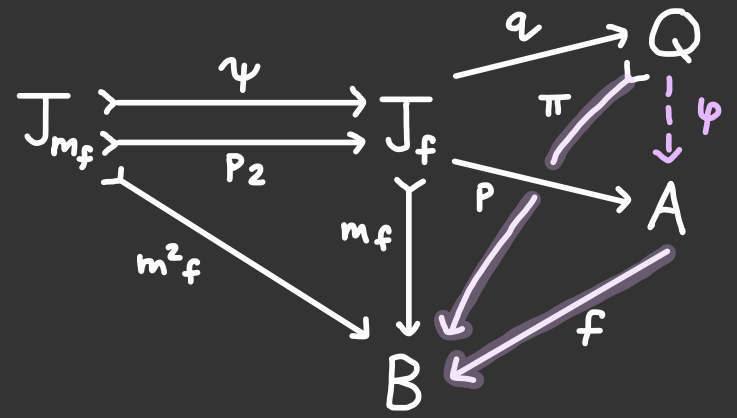


Step 2:  $R\text{-Alg} \longrightarrow vLens(\mathcal{C})$

Assume the coequaliser in  $\mathcal{C}/B$ :



Then we construct:



$\psi \in \mathcal{W}$  since  $p_0$  is a split coequaliser.



# DOUBLE CATEGORY OF COMORPHISMS

Let  $(\mathcal{C}, \mathcal{W}, \mathcal{E}, \mathcal{M})$  be as before,  
and assume  $\mathcal{C}$  has pullbacks\*

The double category  $\mathbb{C}omo(\mathcal{C})$  has:

- Category of objects given by  $\mathcal{C}$ .
- Category of morphisms  $v\mathbb{C}omo(\mathcal{C})$

$$\begin{array}{ccc}
 A & \longrightarrow & A' \\
 \uparrow s & & \uparrow s \\
 X & \longrightarrow & X' \\
 \downarrow & & \downarrow \\
 B & \longrightarrow & B'
 \end{array}$$



The double category  $\mathbb{L}ens(\mathcal{C})$   
is the **completion** of  $\mathbb{C}omo(\mathcal{C})$   
to a right-connected double cat.

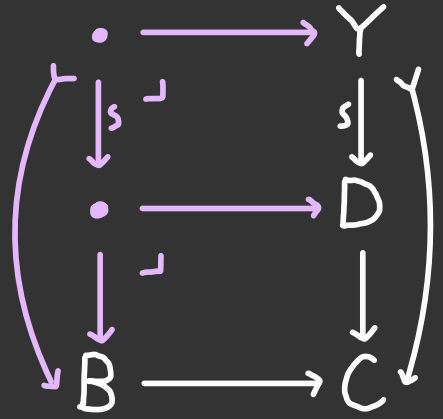
Moreover, if  $\mathcal{C} \longrightarrow \mathit{Disc}(\mathcal{C})$  has  
a right adjoint, then we have  
that  $v\mathbb{L}ens(\mathcal{C}) \longrightarrow v\mathbb{C}omo(\mathcal{C})$  is  
**comonadic** over the cod functor.

**Question:** What can we learn  
from this "duality"?

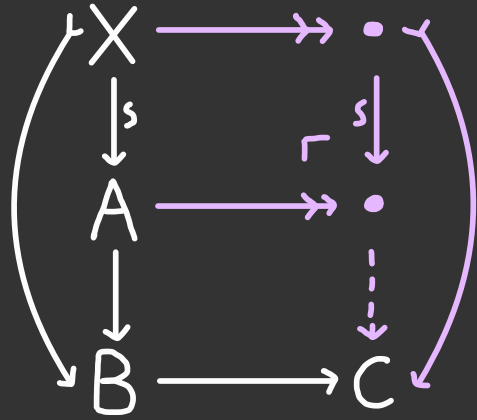
# CHANGE OF BASE

The codomain map of  $\mathbb{L}ens(\mathcal{C})$  given by  $\nu\mathbb{L}ens(\mathcal{C}) \xrightarrow{\text{cod}} \mathcal{C}$  has fibre over  $B \in \mathcal{C}$  denoted  $\mathbb{L}ens(B)$ .

If  $\mathcal{C}$  has pullbacks, then  $\text{cod}$  is a fibration:



If  $\mathcal{C}$  has pushouts, then  $\text{cod}$  is an opfibration:



Thus for every  $B \rightarrow C$  we have:

$$\mathbb{L}ens(B) \xrightleftharpoons{\perp} \mathbb{L}ens(C)$$

and  $\text{cod}$  is a bifibration.

## SUMMARY OF THE TALK

Q: What is the general setting in which lenses should be considered?

A: In a category with pullbacks, pushouts, an idempotent comonad, and an orthogonal factorisation system.

Q: Why is lifting an intrinsic aspect of lenses?

A: Because they are  $R$ -algebras for an AWFS. Lenses are morphisms which are weakly equivalent to the right class of an OFS via an idempotent comonad.

Q: How can we bridge these two approaches to lenses?

A: Using the double category  $\mathbb{L}ens(\mathcal{C})$ .