The double category of generalised lenses

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MOTIVATION

Lenses as morphisms

$$
\begin{tikzcd}
 & & \bullet \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & X \arrow[dl, "\varphi"'] \arrow[dr, "\bar{\varphi}"] & & Y \arrow[dl, "\gamma"'] \arrow[dr, "\bar{\gamma}"] & \\
A \arrow[rr, "f"'] & & B \arrow[rr, "g"'] & & C
\end{tikzcd}
$$

Lenses as objects

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "\varphi"'] \arrow[dd, bend right=60, "\bar{\varphi}"'] & X' \arrow[d, "\varphi'"] \arrow[dd, bend left=60, "\bar{\varphi}'"] \\
A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
B \arrow[r] & B'
\end{tikzcd}
$$

- How can we bridge these two approaches to lenses?
- What is the general setting in which lenses should be considered?
- Why is lifting an intrinsic aspect of lenses?

OVERVIEW OF THE TALK

1. Background and generalised lenses

2. Algebraic weak factorisation systems

$$\begin{tikzcd}
 & & X \arrow[d, "\wr"] \arrow[dd, tail, bend left] \\
C_0 \arrow[r, "\sim"] \arrow[d] \arrow[urr, dashed, bend left] & C \arrow[r] \arrow[d] & A \arrow[d] \\
J \arrow[r, tail] & D \arrow[r] \arrow[ur, dashed] \arrow[ul, phantom, "\ulcorner", very near start] & B
\end{tikzcd}$$

3. The double category of generalised lenses

BACKGROUND

A functor $f : A \to B$ is called:

- **bijective-on-objects** if we have:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & A \arrow[d] \\ \{\bullet\} \arrow[r] \arrow[ur, dashed, "\exists!"] & B \end{tikzcd}$$

- a **discrete opfibration** if we have:

$$\begin{tikzcd} \{\bullet\} \arrow[r] \arrow[d] & A \arrow[d] \\ \{\bullet \to \bullet\} \arrow[r] \arrow[ur, dashed, "\exists!"] & B \end{tikzcd}$$

A **delta lens** $(f, \varphi) : A \to B$ is a commutative diagram in $\mathcal{C}at$,

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\overline{\varphi}"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

where φ is **bijective-on-objects** and $\overline{\varphi}$ is a **discrete opfibration**.

Question: How can we interpret this diagram in other categories?

$\boxed{4}$ IDEMPOTENT COMONADS & DISCRETE OBJECTS

- Let $(-)_0 : \mathcal{C} \to \mathcal{C}$ be an idempotent comonad with counit $i_A : A_0 \to A$.
- An object X is discrete if $X \cong A_0$.
- The full subcategory $\mathrm{Disc}(\mathcal{C})$ on discrete objects is coreflective.
- A morphism $f : A \to B$ is a weak equivalence if f_0 is an iso; will be denoted: $\bullet \xrightarrow{\sim} \bullet$

- The class $\mathrm{Iso}(\mathcal{C}) \subseteq \mathcal{W} \subseteq \mathcal{C}$ of weak equivalences satisfies 2-out-of-3 and is stable under pullback (& pushouts*).
- Universal property:

$$\begin{tikzcd}
 & X \arrow[d, "\wr"] \\
A_0 \arrow[ur, dashed, "\exists!"] \arrow[r] & B
\end{tikzcd}$$

ORTHOGONAL FACTORISATION SYSTEMS

- Let $\mathcal{C}$ be a category t/w classes of morphisms $\mathrm{Iso}(\mathcal{C}) \subseteq \mathcal{E}, \mathcal{M} \subseteq \mathcal{C}$ which are closed under composition.

- The pair $(\mathcal{E}, \mathcal{M})$ is an orthogonal factorisation system on $\mathcal{C}$ if:
  - every morphism decomposes:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e_f \in \mathcal{E}"'] & & B \\ & J_f \arrow[ur, tail, "m_f \in \mathcal{M}"'] & \end{tikzcd}$$

  - the following universal property holds:

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, two heads, "e"'] & \bullet \arrow[d, tail, "m"] \\ \bullet \arrow[r, "v"'] \arrow[ur, dashed, "\exists!"] & \bullet \end{tikzcd}$$

- Several nice properties:
  - if $gf \in \mathcal{M}$ and $g \in \mathcal{M}$, then $f \in \mathcal{M}$
  - morphisms in $\mathcal{M}$ are stable under pullback.

COMORPHISMS & GENERALISED LENSES

- Let $\mathcal{C}$ be a category with,
  - an idempotent comonad $(-)_0$ yielding class $\mathcal{W}$ of weak equiv.
  - an O.F.S. given by $(\mathcal{E}, \mathcal{M})$

  such that $\mathcal{W} \cap \mathcal{M} = \mathrm{Iso}(\mathcal{C})$ and $\mathrm{Disc}(\mathcal{C}) \hookrightarrow \mathcal{M}$.

- A comorphism $\varphi : A \nrightarrow B$ is an isomorphism class of spans:

$$\begin{tikzcd} A & X \arrow[l, "\varphi"', "\sim"] \arrow[r, tail, "\overline{\varphi}"] & B \end{tikzcd}$$

weak equivalence $\mathcal{M}$-morphism

- A generalised lens $(f, \varphi) : A \to B$ is a morphism $f : A \longrightarrow B$ and a comorphism $\varphi : A \nrightarrow B$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\varphi"', "\sim"] \arrow[dr, tail, "\overline{\varphi}"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

Question: Which lifting problems do generalised lenses solve?

LIFTING FOR GENERALISED LENSES

A step-by-step construction:

1. Consider a lifting problem whose right side is a lens
2. Precompose with counit and use universal property of w.e.
3. Factorise the composite and use universal property of O.F.S.
4. Suppose the square is a pushout and use universal property.
5. Rest.

$$
\begin{tikzcd}
C_0 \arrow[rr, dashed, "\exists!"] \arrow[d, equal] & & X \arrow[d, "s"] \arrow[dd, tail, bend left=60] \\
C_0 \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "(*)"] & C \arrow[r] \arrow[d, two heads] & A \arrow[d] \\
J \arrow[r, tail] \arrow[uurr, dashed, bend left, "\exists!"] & D \arrow[r] \arrow[ur, dashed, "\exists!"] \arrow[ul, phantom, "\ulcorner", very near start] & B
\end{tikzcd}
$$

Takeaway: Lenses have the right lifting property against morphisms satisfying (*).

INITIAL FUNCTORS

A functor $f : A \to B$ is called initial if any of the following conditions hold:

1) for all $b \in B$, f/b is connected.
2) for all $g : B \to C$, $\lim(g) \simeq \lim(g \circ f)$.

Examples:

- A functor $1 \to B$ is initial $\Leftrightarrow$ it chooses an initial object in B.
- A functor $A_0 \to B$ is initial $\Leftrightarrow$ it chooses local initial objects in B.
- Left adjoint functors.

The comprehensive factorisation system decomposes every functor:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\text{initial}"'] & & B \\
& \bullet \arrow[ur, "\text{disc. opf.}"'] &
\end{tikzcd}$$

Example: The comprehensive factorisation of $f : A_0 \to B$ is:

$$\begin{tikzcd}
a \arrow[d, mapsto] & A_0 \arrow[rr, "f"] \arrow[dr] & & B & \mathrm{cod}(u) \\
(a, 1_{fa}) & & f/B \arrow[ur] & & (a, u) \arrow[u, mapsto]
\end{tikzcd}$$

UNPACKING LIFTS FOR DELTA LENSES

Consider a lifting problem:

$$\begin{tikzcd} \{*\} \arrow[r, "a"] \arrow[d, "\text{initial obj.}"'] & A \arrow[d, "f"] \\ fa/B \arrow[r, tail, "\mathrm{cod}"'] & B \end{tikzcd}$$

A solution $p : fa/B \longrightarrow A$ states:

- $p(a, 1_{fa}) = a$ & $fp(a,u) = \mathrm{cod}(u)$
- $p\langle 1_a, 1_{fa}\rangle = 1_a$ & $fp\langle 1_a, u\rangle = u$
- $p\langle 1_a, v \circ u\rangle = p\langle 1_a, v\rangle \circ p\langle 1_a, u\rangle$

Then f has a local lens structure at the object $a \in A$.

- The coproduct over all $a \in A$ is a solution, but doesn't give a (global) delta lens structure.
- Conversely, the functors which lift against delta lenses are relatively easy to understand.

$$\begin{tikzcd} C_0 \arrow[r, "i_C"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & C \arrow[d] \\ fi_C/D \arrow[r, tail] & D \end{tikzcd}$$

Paste local initial objects along an object inclusion

10 ALGEBRAIC WEAK FACTORISATION SYSTEMS

An algebraic weak factorisation system (AFWS) on $\mathcal{C}$ consists of:

- A functorial factorisation (L, E, R)

$$\begin{tikzcd}
X \arrow[r, "Lf"] \arrow[rr, bend left, "f"] \arrow[d, "h"'] & Ef \arrow[r, "Rf"] \arrow[d, "E(h,k)"] & Y \arrow[d, "k"] \\
W \arrow[r, "Lg"'] \arrow[rr, bend right, "g"'] & Eg \arrow[r, "Rg"'] & Z
\end{tikzcd}$$

- An extension of (L, ε) to a comonad (L, ε, Δ).

- An extension of (R, η) to a monad (R, η, μ).

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "Lf"'] \arrow[dr, phantom, "\text{counit } \varepsilon_f"] & X \arrow[r, "Lf"] \arrow[d, "f"] \arrow[dr, phantom, "\text{unit } \eta_f"] & Ef \arrow[d, "Rf"] \\
Ef \arrow[r, "Rf"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

- A distributive law $\delta : LR \Rightarrow RL$ of the comonad L over the monad R:

$$\begin{tikzcd}
Ef \arrow[r, "\Delta_f"] \arrow[d, "LRf"'] \arrow[dr, phantom, "\delta_f"] & ELf \arrow[d, "RLf"] \\
ERf \arrow[r, "\mu_f"'] & Ef
\end{tikzcd}$$

THE MONAD FOR GENERALISED LENSES...

Let $(\mathcal{C}, \mathcal{W}, \mathcal{E}, \mathcal{M})$ be as before, and assume $\mathcal{C}$ has pushouts*

$$\begin{tikzcd}
A_0 \arrow[r, "\sim"'] \arrow[d, "e_f"'] & A \arrow[r, equal] \arrow[d, "Lf"] & A \arrow[d, "f"] \\
J_f \arrow[r, "\sim"'] \arrow[rr, bend right, tail, "m_f"'] & Ef \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "Rf"] & B
\end{tikzcd}$$

$$\begin{tikzcd}
(Ef)_0 \arrow[r, "\sim"'] \arrow[d, two heads, "e_{Rf}"'] & Ef \arrow[r, equal] \arrow[d, two heads, "LRf"] & Ef \arrow[d, "Rf"] \\
J_{Rf} \arrow[r, "\sim"'] \arrow[rr, bend right, tail, "m_{Rf}"'] & ERf \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "R^2f"] & B
\end{tikzcd}$$

Since we have,

$$\begin{tikzcd}
(Ef)_0 \arrow[r, "\sim"] \arrow[d, two heads, "e_{Rf}"'] & J_f \arrow[d, tail, "m_f"] \\
J_{Rf} \arrow[r, tail, "m_{Rf}"'] \arrow[ur, dashed, tail, "\exists! \sigma"] & B
\end{tikzcd}$$

the multiplication is given by:

$$\begin{tikzcd}
(Ef)_0 \arrow[r, two heads, "e_{Rf}"] \arrow[d, "s"'] & J_{Rf} \arrow[r, tail, "\sigma"] \arrow[d, "s"] & J_f \arrow[d, "s"] \arrow[dr, tail, "m_f"] & \\
Ef \arrow[r, two heads, "LRf"'] \arrow[rr, bend right, "1"'] & ERf \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, two heads, "\mu_f"] \arrow[rr, bend right, "R^2f"'] & Ef \arrow[r, "Rf"'] & B
\end{tikzcd}$$

... AND THE CORRESPONDING A.W.F.S.

To construct the comonad,

$$\begin{tikzcd}
A_0 \arrow[r, "\sim"'] \arrow[d, "e_{Lf}"'] & A \arrow[r, equal] \arrow[d, "L^2 f"] & A \arrow[d, "Lf"] \\
J_{Lf} \arrow[r, "\sim"'] \arrow[rr, tail, bend right, "m_{Lf}"'] & ELf \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "RLf"] & Ef
\end{tikzcd}$$

note that we have:

$$\begin{tikzcd}
A_0 \arrow[r, two heads, "e_{Lf}"] \arrow[d, "e_f"'] & J_{Lf} \arrow[d, tail, "m_{Lf}"] \\
J_f \arrow[r, "\sim"'] \arrow[ur, dashed, "\tau"'] & Ef
\end{tikzcd}$$

The comultiplication is given by:

$$\begin{tikzcd}
A_0 \arrow[r, two heads, "e_f"] \arrow[d, "s"'] & J_f \arrow[r, two heads, "\tau"] \arrow[d, "s"] & J_{Lf} \arrow[d, "s"] \\
A \arrow[r, two heads, "Lf"'] \arrow[rr, bend right, "L^2 f"'] & Ef \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "\Delta_f"] & ELf
\end{tikzcd}$$

Altogether this gives the data for an AWFS on $\mathcal{C}$, constructed from the OFS $(\mathcal{E}, \mathcal{M})$ and $\mathcal{W}$.

Question: What are the R-algebras?

UNPACKING THE R-ALGEBRAS

Consider the monad (R, η, μ) on $\mathcal{C}^2$.

- A R-algebra on $f : A \to B$ is given by $\hat{p} : Ef \to A$ such that $f \circ \hat{p} = Rf$ and other axioms.

- R-algebras are equivalent to:

$$
\begin{tikzcd}
A_0 \arrow[r, "\sim"] \arrow[d, "e_f"'] & A \arrow[d, "f"] \\
J_f \arrow[ur, "p"] \arrow[r, tail, "m_f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
J_{m_f} \arrow[r, "p_2"] \arrow[d, "\psi"'] & J_f \arrow[d, "p"] \\
J_f \arrow[r, "p"'] & A
\end{tikzcd}
$$

where we have:

$$
\begin{tikzcd}
(J_f)_0 \arrow[r, "\sim"] \arrow[d, "e_{m_f}"'] & J_f \arrow[d, tail, "m_f"] & A_0 \arrow[l, "e_f"'] & (J_f)_0 \arrow[l, "p_0"'] \arrow[d, "e_{m_f}"] \\
J_{m_f} \arrow[ur, dashed, tail, "\psi"] \arrow[r, tail, "m_f^2"'] & B & & J_{m_f} \arrow[ll, tail, "m_f^2"] \arrow[ull, dashed, tail, "p_2"']
\end{tikzcd}
$$

For any AWFS on $\mathcal{C}$, there is a double category $\mathbb{R}\text{-}\mathbb{A}\mathrm{lg}$ whose:

- Category of objects is $\mathcal{C}$.
- Category of morphisms is the category of R-algebras.

Claim: The double cat of lenses is isomorphic to $\mathbb{R}\text{-}\mathbb{A}\mathrm{lg}$.

$\boxed{14}$ DOUBLE CATEGORY OF GENERALISED LENSES

Let $(\mathcal{C}, \mathcal{W}, \mathcal{E}, \mathcal{M})$ be as before, and assume $\mathcal{C}$ has pullbacks*

The double category $\mathbb{L}\mathrm{ens}(\mathcal{C})$ has:

- Category of objects given by $\mathcal{C}$.
- Category of morphisms $\mathrm{vLens}(\mathcal{C})$

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "\varphi"', "s"] \arrow[dd, tail, bend right=60, "\bar{\varphi}"'] & X' \arrow[d, "s"', "\varphi'"] \arrow[dd, tail, bend left=60, "\bar{\varphi}'"] \\
A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
B \arrow[r] & B'
\end{tikzcd}
$$

The double category $\mathbb{L}\mathrm{ens}(\mathcal{C})$ is right-connected since we have cells for each lens.

$$
\begin{tikzcd}
X \arrow[r, tail, "\bar{\varphi}"] \arrow[d, "\varphi"', "s"] & B \arrow[d, equal] \\
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

For $\mathbb{L}\mathrm{ens}(\mathcal{C}) \cong R\text{-}\mathbb{A}\mathrm{lg}$, we need to show that $\mathrm{vLens}(\mathcal{C}) \xrightarrow{\text{monadic}} \mathcal{C}^{2}$.

MONADICITY

Let (R, η, μ) be the monad on $\mathcal{C}^2$ as before. We want $\mathrm{vLens}(\mathcal{C})$ to be isomorphic to R-Alg.

Step 1: $\mathrm{vLens}(\mathcal{C}) \longrightarrow R\text{-}\mathrm{Alg}$

$$
\begin{tikzcd}
 & & X \arrow[d, "s"', "\varphi"] \arrow[dd, tail, bend left=60, "\bar{\varphi}"] \\
A_0 \arrow[urr, bend left, "\exists!"] \arrow[rr, "\sim"] \arrow[d, "e_f"'] & & A \arrow[d, "f"] \\
J_f \arrow[uurr, dashed, "\exists!"] \arrow[urr, "p"'] \arrow[rr, tail, "m_f"'] & & B
\end{tikzcd}
$$

Step 2: $R\text{-}\mathrm{Alg} \longrightarrow \mathrm{vLens}(\mathcal{C})$

Assume the coequaliser in $\mathcal{C}/B$:

$$
\begin{tikzcd}
J_{m_f} \arrow[rr, tail, shift left, "\psi"] \arrow[rr, tail, shift right, "p_2"'] \arrow[dr, tail, "m_f^2"'] & & J_f \arrow[r, dashed, "q"] \arrow[dl, tail, "m_f"] & Q \arrow[dll, bend left, "\pi"] \\
 & B & &
\end{tikzcd}
$$

$\pi \in \mathcal{M}$

Then we construct:

$$
\begin{tikzcd}
 & & & Q \arrow[d, dashed, "\varphi"] \arrow[ddl, tail, "\pi"'] \\
J_{m_f} \arrow[rr, tail, shift left, "\psi"] \arrow[rr, tail, shift right, "p_2"'] \arrow[drr, tail, "m_f^2"'] & & J_f \arrow[ur, "q"] \arrow[r, "p"] \arrow[d, tail, "m_f"] & A \arrow[dl, "f"] \\
 & & B &
\end{tikzcd}
$$

$\varphi \in \mathcal{W}$ since p_0 is a split coequaliser.

DOUBLE CATEGORY OF COMORPHISMS

Let $(\mathcal{C}, \mathcal{W}, \mathcal{E}, \mathcal{M})$ be as before, and assume $\mathcal{C}$ has pullbacks*

The double category $\mathbb{C}\mathrm{omo}(\mathcal{C})$ has:

- Category of objects given by $\mathcal{C}$.
- Category of morphisms $v\mathcal{C}\mathrm{omo}(\mathcal{C})$

$$\begin{tikzcd} A \arrow[r] & A' \\ X \arrow[u, "s"] \arrow[r] \arrow[d, tail] & X' \arrow[u, "s"'] \arrow[d, tail] \\ B \arrow[r] & B' \end{tikzcd}$$

The double category $\mathbb{L}\mathrm{ens}(\mathcal{C})$ is the completion of $\mathbb{C}\mathrm{omo}(\mathcal{C})$ to a right-connected double cat.

Moreover, if $\mathcal{C} \longrightarrow \mathrm{Disc}(\mathcal{C})$ has a right adjoint, then we have that $v\mathrm{Lens}(\mathcal{C}) \longrightarrow v\mathcal{C}\mathrm{omo}(\mathcal{C})$ is comonadic over the cod functor.

Question: What can we learn from this "duality"?

CHANGE OF BASE

The codomain map of $\mathbb{L}\mathrm{ens}(\mathcal{C})$ given by $\mathrm{vLens}(\mathcal{C}) \xrightarrow{\mathrm{cod}} \mathcal{C}$ has fibre over $B \in \mathcal{C}$ denoted $\mathcal{L}\mathrm{ens}(B)$.

If $\mathcal{C}$ has pullbacks, then cod is a fibration:

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "s"] \arrow[dd, bend right=60] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "s"'] \arrow[dd, bend left=60] \\
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}
$$

If $\mathcal{C}$ has pushouts, then cod is an opfibration:

$$
\begin{tikzcd}
X \arrow[r, two heads] \arrow[d, "s"] \arrow[dd, bend right=60] & \bullet \arrow[d, "s"'] \arrow[dd, bend left=60] \\
A \arrow[r, two heads] \arrow[d] \arrow[ur, phantom, "\ulcorner", very near end] & \bullet \arrow[d, dashed] \\
B \arrow[r] & C
\end{tikzcd}
$$

Thus for every $B \longrightarrow C$ we have:

$$
\begin{tikzcd}
\mathcal{L}\mathrm{ens}(B) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{L}\mathrm{ens}(C) \arrow[l, shift left]
\end{tikzcd}
$$

and cod is a bifibration.

SUMMARY OF THE TALK

Q: What is the general setting in which lenses should be considered?

A: In a category with pullbacks, pushouts, an idempotent comonad, and an orthogonal factorisation system.

Q: Why is lifting an intrinsic aspect of lenses?

A: Because they are R-algebras for an AWFS. Lenses are morphisms which are weakly equivalent to the right class of an OFS via an idempotent comonad.

Q: How can we bridge these two approaches to lenses?

A: Using the double category $\mathbb{L}\mathrm{ens}(\mathcal{C})$.