Delta lenses as coalgebras for a comonad

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OVERVIEW OF THE TALK

Delta lens

Get functor

$$A \xrightarrow{f} B$$

algebraic $\rightsquigarrow$

$\leftsquigarrow$ coalgebraic

Put cofunctor

$$A \xleftarrow{\varphi} X \xrightarrow{\bar{f}} B$$

- Every lens consists of two parts: Get (forwards) and Put (backwards).

- Usually a d-lens is understood as a Get functor equipped with additonal algebraic structure specifying the Put.

- Here we take a Put-based approach to d-lenses as coalgebras for a comonad.

HISTORY & MOTIVATION

- 2012: Gibbons & Johnson compare (co)-algebraic approaches to classical state-based lenses, internalise to any C.C.C.
- 2013: Johnson & Rosebrugh show d-lenses are algebras for a semi-monad.
- 2016: Ahman & Uustalu establish a construction assigning every cofunctor (morphism of directed containers) to a d-lens.
- 2017: Ahman & Uustalu characterise d-lenses as cofunctors with additonal structure, given by a functor.
- 2020: A diagrammatic characterisation of d-lenses is introduced.

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REVIEWING D-LENSES & COFUNCTORS

- A d-lens $(f,\varphi): A \nrightarrow B$ consists of a a functor $f: A \to B$ and a lifting operation,

$$\begin{tikzcd} A \arrow[d, "{(f,\varphi)}"'] & a \arrow[r, "{\varphi(a,u)}"] \arrow[d, dotted, no head] & a' \arrow[d, dotted, no head] \\ B & fa \arrow[r, "u"] & b \end{tikzcd}$$

satisfying the axioms:

(1) $f\varphi(a,u) = u$

(2) $\varphi(a, 1_{fa}) = 1_a$

(3) $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a,u)$

- A cofunctor $(f_0,\varphi): A \nrightarrow B$ consists of a function $f_0 : Ob(A) \to Ob(B)$ and a lifting operation,

$$\begin{tikzcd} A \arrow[d, "{(f_0,\varphi)}"'] & a \arrow[r, "{\varphi(a,u)}"] \arrow[d, dotted, no head] & a' \arrow[d, dotted, no head] \\ B & f_0 a \arrow[r, "u"] & b \end{tikzcd}$$

satisfying the axioms:

(1) $f_0\, cod(\varphi(a,u)) = cod(u)$

(2) $\varphi(a, 1_{fa}) = 1_a$

(3) $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a,u)$

DIAGRAMMATIC REPRESENTATIONS USING FUNCTORS

- A functor $f : A \to B$ is called:
  - * **bijective-on-objects** if $f_0 : \mathrm{Ob}(A) \to \mathrm{Ob}(B)$ is a bijection.
  - * **discrete opfibration** if for all pairs $(a \in A,\ u : fa \to b \in B)$ there is a unique morphism $w : a \to a'$ such that $fw = u$.

$$
\begin{tikzcd}
A \arrow[d, "f"'] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, dashed, "\exists!\, w"] \arrow[d, dotted, no head] & a' \arrow[d, dotted, no head] \\
fa \arrow[r, "u"] & b
\end{tikzcd}
$$

- A **cofunctor** $(f_0, \varphi) : A \nrightarrow B$ is the same as a span of functors,

$$
\begin{tikzcd}
A & X \arrow[l, "\varphi"'] \arrow[r, "\bar{f}"] & B
\end{tikzcd}
$$

where φ is bijective-on-objects and $\bar{f}$ is a discrete opfibration.

- A **d-lens** $(f, \varphi) : A \nrightarrow B$ is the same same as a commutative diagram:

$$
\begin{tikzcd}
& X \arrow[dl, "\varphi"'] \arrow[dr, "\bar{f}"] & \\
A \arrow[rr, "f"] & & B
\end{tikzcd}
$$

CATEGORY OF COFUNCTORS OVER A BASE

- Let B be a fixed category (view).
- Let $Cof(B)$ be a category whose:
  * objects are cofunctors into B;
  * morphisms are given by

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "{(f_0, \varphi)}"'] & & C \arrow[dl, "{(g_0, \gamma)}"] \\ & B & \end{tikzcd}$$

functors h such that

$f_0 a = g_0 h a$ and $h\varphi(a,u) = \gamma(ha, u)$.

- Idea: morphisms are functors between source categories which preserve the chosen lifts.
- Morphisms are equivalent to:

$$\begin{tikzcd} A \arrow[rr, "h"] & & C \\ X \arrow[u, "\varphi"] \arrow[rr, "\bar{h}"] \arrow[dr, "\bar{f}"'] & & Y \arrow[u, "\gamma"'] \arrow[dl, "\bar{g}"] \\ & B & \end{tikzcd}$$

where $\bar{h}$ is an induced functor.

D-LENSES AS MORPHISMS BETWEEN COFUNCTORS

- Let $1_B : B \nrightarrow B$ be the trivial cofunctor on B
- Proposition: Morphisms in $\mathcal{C}of(B)$ to the trivial cofunctor are equivalent to d-lenses into B.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "{(f_0, \varphi)}"', "|"] & & B \arrow[dl, "1_B", "|"'] \\ & B & \end{tikzcd}$$

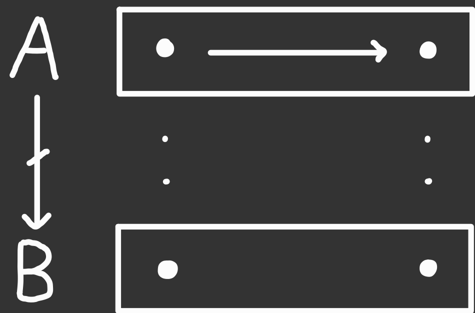
$\simeq$ functors f such that $f_0 a = f a$ and $f\varphi(a,u) = \pi(fa, u) = u$.

- Define the category $\mathcal{L}ens(B)$ of d-lenses over a base B as the slice category $\mathcal{C}of(B)/1_B$.
- There is a forgetful functor $L : \mathcal{L}ens(B) \longrightarrow \mathcal{C}of(B)$ given by:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, equal] & & A \\ X \arrow[u, "\varphi"] \arrow[r, "\bar{f}"] \arrow[dr, "\bar{f}"'] & B \arrow[dl, equal] & \arrow[r, squiggly, "L"] & X \arrow[u, "\varphi"'] \arrow[d, "\bar{f}"'] \\ & B & & B \end{tikzcd}$$

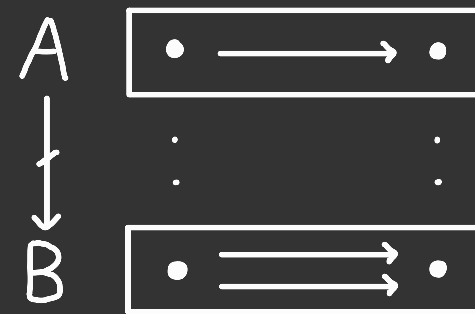
WHY ISN'T EVERY COFUNCTOR A D-LENS?

- There are two primary obstructions which could occur:



(1) Not enough morphisms in view.

Solution: delete morphisms in A.



(2) Too many morphisms in view.

Solution: duplicate morphisms in A.

- The key to constructing a universal solution is to use the codiscrete category $\hat{B}$ which has a unique morphism between each object in $Ob(B)$.

A RIGHT ADJOINT: THE COFREE D-LENS ON A COFUNCTOR

- **Proposition:** There is an adjunction,

$$\begin{tikzcd} \mathcal{L}\mathrm{ens}(B) \arrow[r, shift left=2, "L"] \arrow[r, phantom, "\perp"] & \mathrm{Cof}(B) \arrow[l, shift left=2, "R"] \end{tikzcd}$$

whose right adjoint sends a cofunctor $(f_0, \varphi) : A \nrightarrow B$ to the cofree d-lens $P \nrightarrow B$ given by:

$$\begin{tikzcd} X \arrow[r, dashed, "\langle \varphi, \bar{f} \rangle"] \arrow[rr, bend left, "\bar{f}"] \arrow[dr, "\varphi"'] & P \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\eta_B"] \\ & A \arrow[r, "\hat{f}_0 \circ \eta_A"'] & \hat{B} \end{tikzcd}$$

- The category P has:
  - * objects the same as A,
  - * morphisms $(w : a \to a', u : fa \to fa')$.

- The **counit** is a morphism in $\mathrm{Cof}(B)$ given by:

$$\begin{tikzcd} P \arrow[rr, "\pi_A"] & & A \\ X \arrow[u, "\langle \varphi, \bar{f} \rangle"] \arrow[rr, equal] \arrow[dr, "\bar{f}"'] & & X \arrow[u, "\varphi"'] \arrow[dl, "\bar{f}"] \\ & B & \end{tikzcd}$$

THE MAIN RESULT

Theorem: The forgetful functor $\mathcal{L}\mathrm{ens}(B) \xrightarrow{L} \mathrm{Cof}(B)$ is comonadic

Proof: Consider a coalgebra for the comonad LR on $\mathrm{Cof}(B)$:

coalgebra map

$$
\begin{tikzcd}
A \arrow[r, "\langle 1_A, f\rangle"] & P \arrow[r, "\pi_A"] & A \\
X \arrow[u, "\varphi"] \arrow[r, equal] \arrow[dr, "\bar{f}"'] & X \arrow[u, "\langle \varphi, \bar{f}\rangle"'] \arrow[r, equal] \arrow[d, "\bar{f}"] \arrow[ur, phantom, "\text{counit}"] & X \arrow[u, "\varphi"'] \arrow[dl, "\bar{f}"] \\
& B &
\end{tikzcd}
$$

- The coalgebra maps <u>into</u> the pullback P, and compatibility with the counit gives $\langle 1_A, f\rangle : A \to P$.
- A coalgebra is a functor $f : A \to B$ such that $f \circ \varphi = \bar{f}$; equivalently $f_0 a = fa$ and $f\varphi(a,u) = u$.

$$\mathcal{L}\mathrm{ens}(B) \simeq \mathrm{Coalg}(LR)$$

SUMMARY OF THE TALK

- Many approaches to d-lenses:
  - A functor and a lifting operation:

$$\begin{tikzcd} A \arrow[d, "{(f,\varphi)}"'] & a \arrow[r, "{\varphi(a,u)}"] \arrow[d, dotted, no head] & a' \arrow[d, dotted, no head] \\ B & fa \arrow[r, "u"] & b \end{tikzcd}$$

  - A certain commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\bar{f}"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

  - An object in the category $\mathrm{Cof}(B)/1_B$

- Our main result gives a Put-based approach: d-lenses into B are coalgebras for a comonad on Cof(B).
- There are nice categorical consequences including:
  - Computing colimits in $\mathcal{L}\mathrm{ens}(B)$
  - Every d-lens factorises through a cofree lens by a b.o.o. functor.
- This result also unifies several previous results in the literature.