

LENSES AS COALGEBRAS  
FOR A COMONAD

BRYCE CLARKE

AUSTRALIAN CATEGORY SEMINAR  
28 APRIL 2021

1

## HISTORY & MOTIVATION

- Classical **state-based lenses** are equivalent to coalgebras for the comonad generated by the adjunction  $(-) \times B \dashv [B, -]$  on  $\text{Set}$ .
- In 2017, Ahman and Uustalu showed explicitly how **(delta) lenses** can be understood as **cofunctors with additional structure**.
- In 2016, [AU] describe a construction on cofunctors which yields lenses.
- Last year I showed that:
  - $\text{Lens}(B) \longrightarrow \text{Cat}/B$  is monadic (**lenses as algebras for a monad**)
  - $\text{Lens}(B) \simeq [B, s/\text{Mult}]_{\text{lax}}$  (**lenses as lax double functors**)

2

## OVERVIEW OF THE TALK

- The central goal is to show that the forgetful functor,

$$\mathbf{Lens}(B) \longrightarrow \mathbf{Cof}(B)$$



is comonadic.

- We will see that this result also holds when the above diagrams are interpreted in a setting more general than  $\mathbf{Cat}$ .

3

## BACKGROUND: BASIC NOTIONS

- A functor  $f:A \rightarrow B$  is **bijective-on-objects** if its underlying object assignment  $f_0:A_0 \rightarrow B_0$  is a bijection.
- A functor  $f:A \rightarrow B$  is a **discrete opfibration** if for all pairs  $(a \in A, u:fa \rightarrow b \in B)$  there exists a unique morphism  $w:a \rightarrow a'$  in  $A$  such that  $fw = u$ .

### Some motivating questions

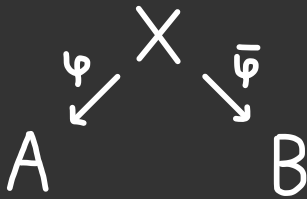
- 1) How might we consider morphisms like discrete opfibrations?
- 2) Can we define these notions in arbitrary categories?

4

## BACKGROUND: COFUNCTORS & LENSES

• Both cofunctors and lenses are like discrete opfibrations.

• A **cofunctor**  $\psi: A \dashrightarrow B$  is a span of functors,

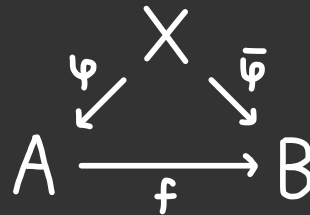


where:

$\psi$  is **bijective-on-objects**;

$\bar{\psi}$  is a **discrete opfibration**.

• A **lens**  $(f, \psi): A \rightleftarrows B$  is a functor  $f: A \rightarrow B$  together with a cofunctor  $\psi: A \dashrightarrow B$  such that the following diagram commutes:



5

## THE SPECIFIC SETTING

Let  $\mathcal{C}at$  be the category of small categories and functors.

- $\mathbf{Set}$  is a reflective subcategory of  $\mathcal{C}at$ :

$$\begin{array}{ccc}
 & \xleftarrow{(-)_o} & \\
 \mathbf{Set} & \xleftarrow{\perp} & \mathcal{C}at \\
 & \xrightarrow{(\hat{-})} & \\
 & & \text{codiscrete category}
 \end{array}$$

underlying objects

- **Bijjective-on-objects functors** form the class of weak equivalences with respect to  $(-)_o: \mathcal{C}at \longrightarrow \mathbf{Set}$ .
- **Discrete opfibrations** form a chosen class of morphisms called "fibrations."

6

## THE GENERAL SETTING

Let  $\mathcal{E}$  be a category with pullbacks equipped with:

- a **reflective subcategory**  $\mathcal{D}$  whose reflector preserves pullbacks;

$$\begin{array}{ccc} & \xleftarrow{(-)_\circ} & \\ \mathcal{D} & \xleftrightarrow[\hat{(-)}]{\perp} & \mathcal{E} \end{array}$$

- the wide subcategory  $\text{Core}(\mathcal{E}) \subseteq \mathcal{W} \subseteq \mathcal{E}$  of **weak equivalences** whose members are sent to isomorphisms by the reflector;

$$\xrightarrow{\sim} \in \mathcal{W}$$

- a wide subcategory  $\text{Core}(\mathcal{E}) \subseteq \text{Fib} \subseteq \mathcal{E}$  of **fibrations**.\*

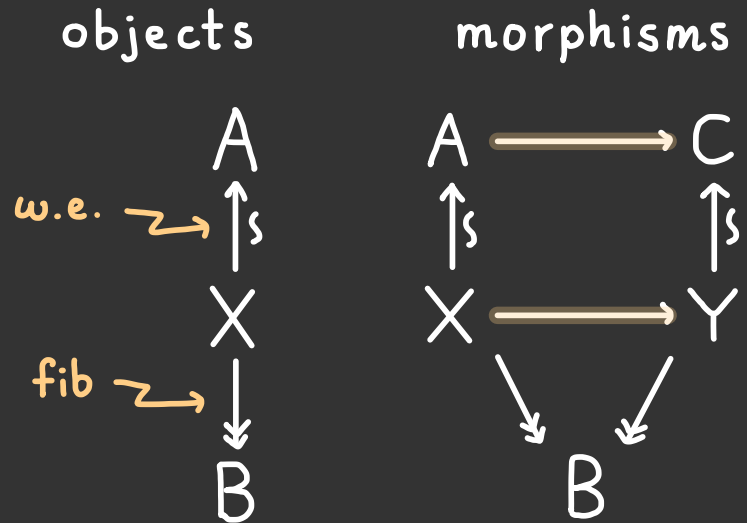
$$\longrightarrow \in \text{Fib}$$

7

## WORKING OVER A BASE

- Let  $\mathcal{E} = (\mathcal{E}, \mathcal{D}, \mathcal{W}, \text{Fib})$  be our setting for the talk.
- Choose an object  $B \in \mathcal{E}$ .
- Let  $\text{Fib}(B)$  be the full subcategory of  $\mathcal{E}/B$  on the class of fibrations (think  $\text{DOpf}(B)$  in  $\text{Cat}$ ).

- Let  $\text{XFib}(B)$  be the category of extended fibrations over  $B$  (think  $\text{Cof}(B)$  in  $\text{Cat}$ ).





## AN EASY LEMMA

Lemma:  $\text{Fib}(B)$  is a coreflective subcategory of  $\text{XFib}(B)$ :

$$\text{Fib}(B) \begin{array}{c} \longleftarrow \\ \text{T} \\ \longrightarrow \end{array} \text{XFib}(B)$$

↙ call the inclusion  $T$

Proof: The counit for the adjunction is given by:

$$\begin{array}{ccc} X & \xrightarrow{\sim} & A \\ \parallel & & \uparrow \wr \\ X & \xlongequal{\quad} & X \\ & \searrow & \swarrow \\ & & B \end{array}$$

9

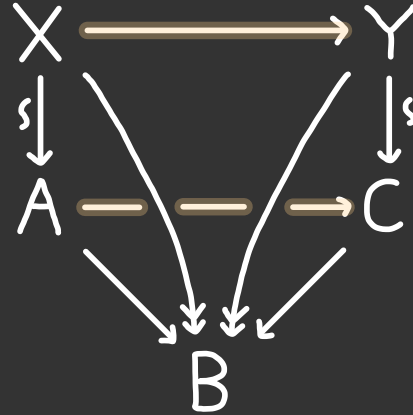
# LENSES OVER A BASE

Let  $\mathcal{L}ens(B)$  be the category of lenses over  $B$  consisting of:

objects



morphisms

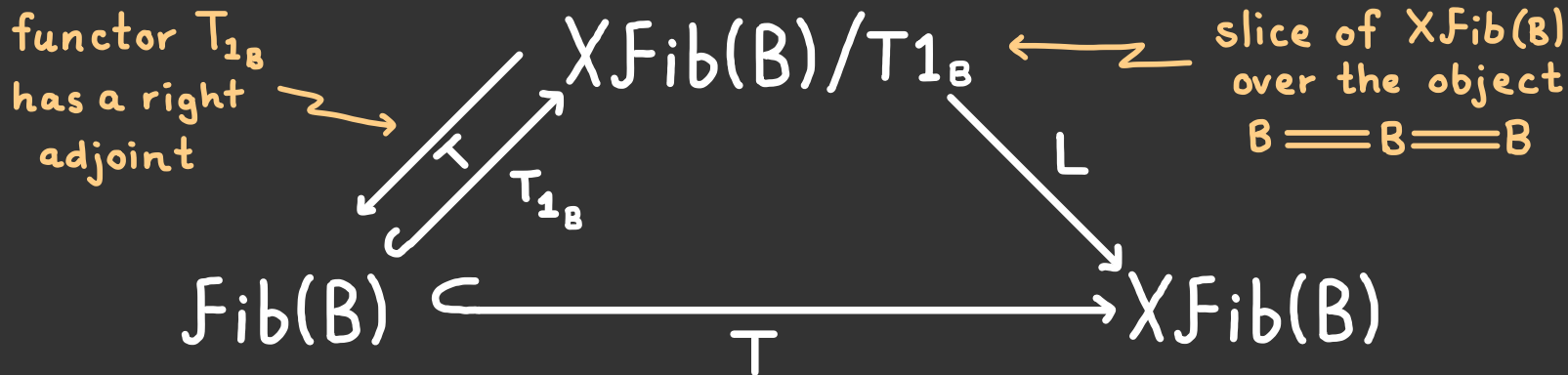


Goal: Show that the forgetful functor  $\mathcal{L}ens(B) \xrightarrow{L} \mathcal{X}Fib(B)$  is comonadic.

10

## ASIDE: AN INTERESTING DIAGRAM

- Since  $\text{Fib}(B)$  has a terminal object  $1_B$ , but  $\text{XFib}(B)$  does not, we can consider the following factorisation:



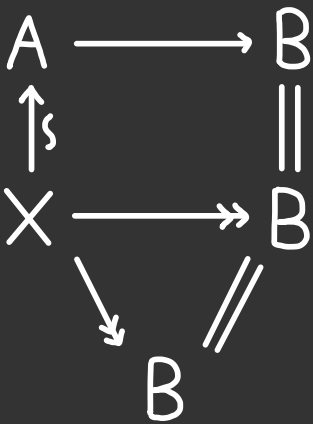
- Is the category  $\text{XFib}(B)/T_{1_B}$  something familiar? Yes!

1 1

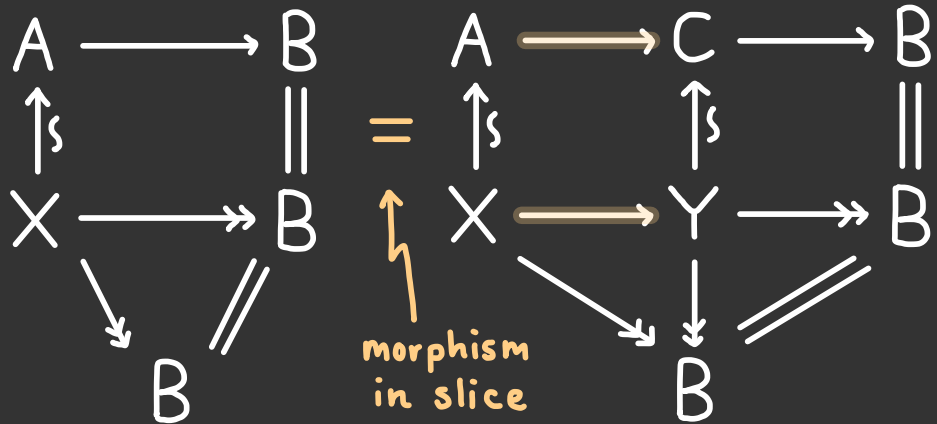
# ASIDE: ANOTHER CHARACTERISATION OF LENSES

Proposition: There is an isomorphism  $\text{Lens}(B) \cong \text{XFib}(B)/T1_B$ .

Proof: Examining the objects and morphisms of  $\text{XFib}(B)/T1_B$ :



Objects are equivalent to lenses

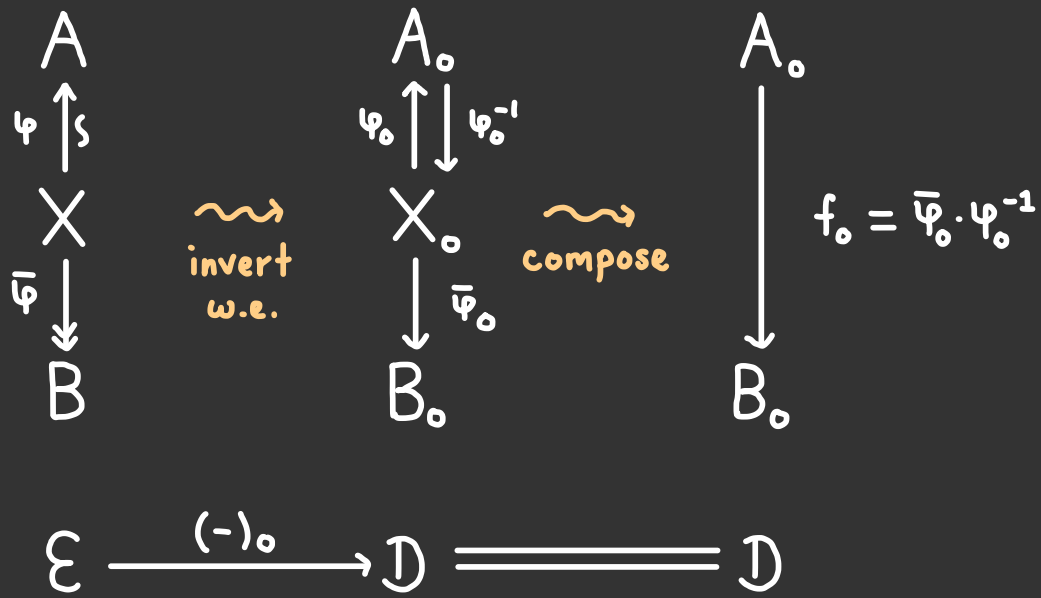


Morphisms are equivalent to those in  $\text{Lens}(B)$

# UNDERLYING OBJECTS

Lemma: There is a functor  $X \text{Fib}(B) \rightarrow \mathcal{D}/B_0$ .

Proof: On objects the functor is given by:



# CONSTRUCTING A RIGHT ADJOINT

Lemma: There is a functor  $X\text{Fib}(B) \xrightarrow{R} \text{Lens}(B)$  given by:

$$\begin{array}{ccc}
 & X & \\
 \varphi \swarrow \wr & & \searrow \bar{\varphi} \\
 A & & B
 \end{array}
 \quad \longrightarrow \quad
 \begin{array}{ccc}
 & X & \\
 \langle \varphi, \bar{\varphi} \rangle \swarrow \wr & & \searrow \bar{\varphi} \\
 P & \xrightarrow{\pi_B} & B
 \end{array}$$

Proof: Consider the pullback:

$$\begin{array}{ccccc}
 & & \bar{\varphi} & & \\
 & & \curvearrowright & & \\
 X & \xrightarrow{\langle \varphi, \bar{\varphi} \rangle} & P & \xrightarrow{\pi_B} & B \\
 \sim & \wr & \downarrow \pi_A & \lrcorner & \downarrow \eta_B \\
 & \varphi & A & \xrightarrow{\hat{f}_0 \circ \eta_A} & \hat{B}_0
 \end{array}$$

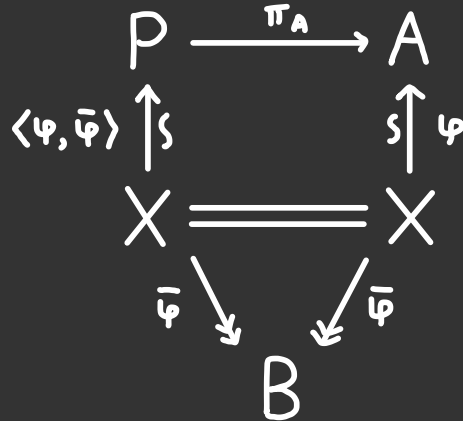
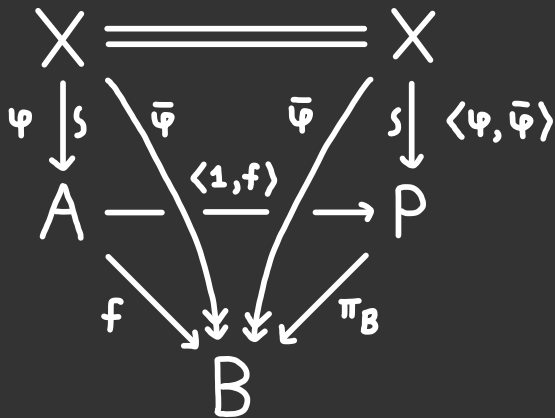
- $\eta$  is the unit of  $(-)_0 \dashv (\hat{-})$  and components are in  $\mathcal{W}$  since counit = id.
- $\pi_A \in \mathcal{W}$  since  $(-)_0$  preserves pullbacks.
- Outer square commutes (abst. nons.)
- $\langle \varphi, \bar{\varphi} \rangle \in \mathcal{W}$  by 2-out-of-3.

# THE MAIN ADJUNCTION

Lemma: There is an adjunction:

$$\text{Lens}(B) \begin{array}{c} \xleftarrow{R} \\ \xrightarrow{T} \\ \xrightarrow{L} \end{array} X\text{Fib}(B)$$

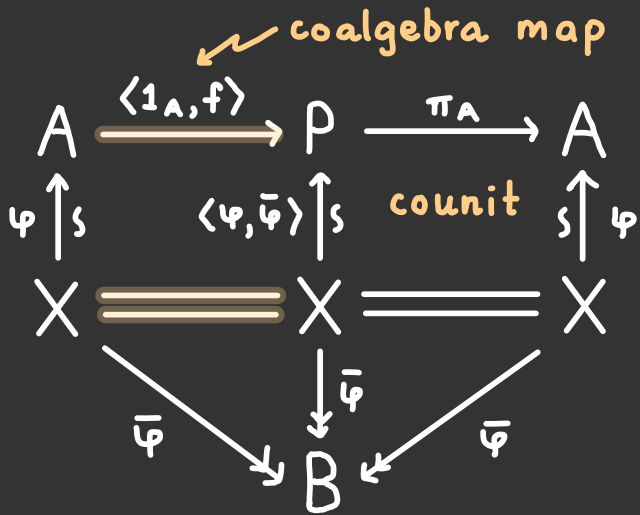
Proof: The unit and counit are given by:



# THE MAIN RESULT

**Theorem:** The forgetful functor  $\text{Lens}(B) \xrightarrow{L} \text{XFib}(B)$  is comonadic

**Proof:** Consider a coalgebra for the comonad  $LR$  on  $\text{XFib}(B)$  :



- The coalgebra maps into the pullback  $P$ , and compatibility with the counit gives  $\langle 1_A, f \rangle: A \rightarrow P$ .
- Thus a coalgebra is a functor  $f: A \rightarrow B$  with  $f \circ \varphi = \bar{\psi}$  and  $f \circ \psi = \bar{\psi}$ .
- Comultiplication adds nothing more.

$$\text{Lens}(B) \simeq \text{Coalg}(LR)$$



## IMMEDIATE COROLLARIES

Let  $\mathcal{E} = \text{Cat}$ ,  $\mathcal{D} = \text{Set}$ , and  $\mathcal{W} = \{\text{bijective-on-objects functors}\}$ .

- When  $\text{Fib} = \{\text{discrete opfibrations}\}$ , then we have that  $\text{Lens} \rightarrow \text{Cof}(B)$  is comonadic (this was our main goal).
- When  $\text{Fib} = \{\text{surjective-on-objects discrete opfibrations}\}$ , then our coalgebras are surjective-on-objects lenses
- In general,  $\text{Fib}$  is any wide subcategory of  $\mathcal{E}$ :
  - What are some other interesting choices for  $\text{Fib}$ ?
  - How can we choose  $\text{Fib}$  to behave like discrete opfibrations?

## FIBRATIONS VIA COMONADS

Let  $\mathcal{E}$  be a category with pullbacks equipped with a comonad  $(D, \varepsilon, \delta)$  which preserves pullbacks.

There is a wide subcategory  $\text{Core}(\mathcal{E}) \subseteq D\text{-Fib} \subseteq \mathcal{E}$  of  $D$ -fibrations consisting of morphisms  $f: A \rightarrow B$  in  $\mathcal{E}$  such that the square

$$\begin{array}{ccc}
 DA & \xrightarrow{Df} & DB \\
 \varepsilon_A \downarrow & & \downarrow \varepsilon_B \\
 A & \xrightarrow{f} & B
 \end{array}$$

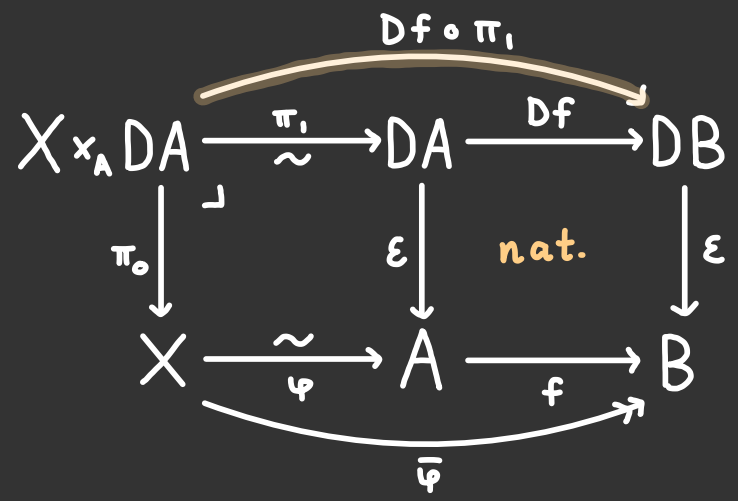
is a pullback.

encodes the universal property  $\rightsquigarrow$

Example: If  $\mathcal{E} = \text{Cat}$  and  $D$  is the *décalage* comonad, then  $D\text{-Fib}$  is the category of *discrete opfibrations*.

# D-UNIVERSAL LENSES

A lens  $(f, \varphi): A \rightleftarrows B$  in a category  $\mathcal{E} = (\mathcal{E}, \mathcal{D}, \mathcal{W}, D\text{-Fib})$  is **D-universal** if the morphism  $Df \circ \pi_1$  is a D-fibration.



**Example:** If  $\mathcal{E} = \text{Cat}$  and  $D$  is the décalage comonad, then a lens is **D-universal** if and only if it is a **split opfibration**.

## REINDEXING FUNCTORS

- Given a lens  $(g, \gamma): B \rightarrow C$ , there is a functor,

$$\Sigma_{(g, \gamma)} : \mathcal{L}ens(B) \rightarrow \mathcal{L}ens(C)$$

defined by post-composition (assuming  $\mathcal{F}ib$  is nice).

- Given a functor  $g: B \rightarrow C$ , there is a functor,

$$\Delta_g : \mathcal{L}ens(C) \rightarrow \mathcal{L}ens(B)$$

defined by pullback (also assuming  $\mathcal{F}ib$  is nice).

- When  $g \in \mathcal{F}ib$ , there is an adjunction  $\Sigma_g \dashv \Delta_g$ .

- Do these functors yield a double category perspective?

## CONNECTION TO THE ALGEBRA STORY?

- What conditions on  $\mathcal{E}$  are required for the functor,

$$\mathcal{L}ens(B) \longrightarrow \mathcal{E}/B$$

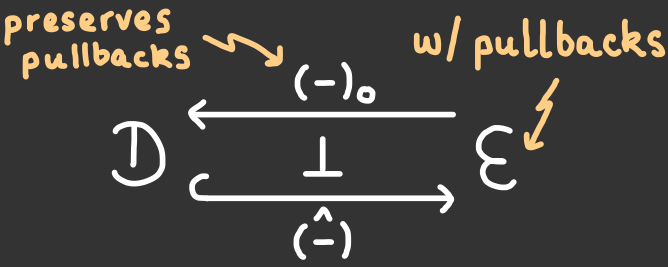
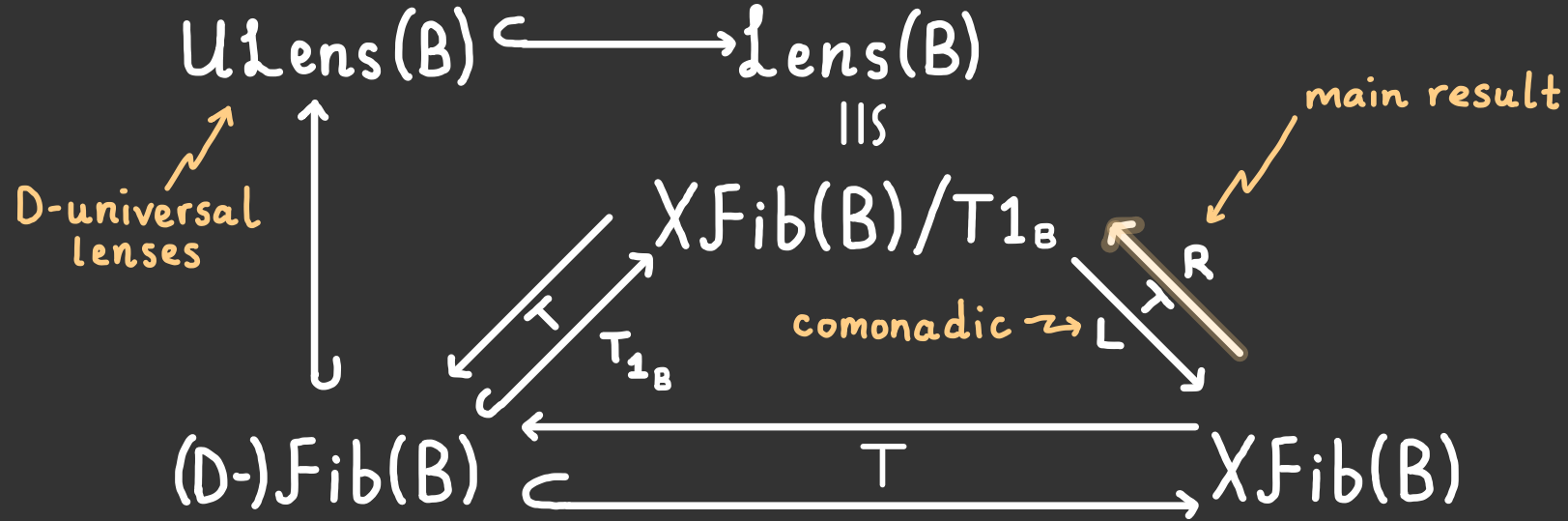
to be **monadic**? (Recall that  $\mathcal{E} = \mathcal{C}at$  is sufficient)

- What can we understand from the composite adjunction?

$$\mathcal{C}of(B) \begin{array}{c} \xleftarrow{\text{comonadic}} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathcal{L}ens(B) \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\text{monadic}} \end{array} \mathcal{C}at/B$$

- We know that  $\mathcal{S}Opf(B) \longrightarrow \mathcal{C}at/B$  is monadic; is the functor  $\mathcal{S}Opf(B) \longrightarrow \mathcal{C}of(B)$  also comonadic?

# SUMMARY OF THE TALK



$W$ -class of weak equivalences w.r.t.  $(-)_o$   
 $Fib$ -class of fibrations (possibly w.r.t.  $D$ )  
 $D$ -comonad on  $E$  preserving pullbacks