Lenses as coalgebras for a comonad

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HISTORY & MOTIVATION

- Classical state-based lenses are equivalent to coalgebras for the comonad generated by the adjunction $(-) \times B \dashv [B, -]$ on Set.
- In 2017, Ahman and Uustalu showed explicitly how (delta) lenses can be understood as cofunctors with additional structure.
- In 2016, [AU] describe a construction on cofunctors which yields lenses.
- Last year I showed that:
  - $\mathrm{Lens}(B) \longrightarrow \mathrm{Cat}/B$ is monadic (lenses as algebras for a monad)
  - $\mathrm{Lens}(B) \simeq [B, s\mathcal{M}\mathrm{ult}]_{lax}$ (lenses as lax double functors)

OVERVIEW OF THE TALK

- The central goal is to show that the forgetful functor,

$$\mathcal{L}\mathrm{ens}(B) \longrightarrow \mathcal{C}\mathrm{of}(B)$$

$$\begin{tikzcd}[column sep=small]
 & X \arrow[dl, "\varphi"'] \arrow[dr, "\overline{\varphi}"] & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
\qquad \longmapsto \qquad
\begin{tikzcd}[column sep=small]
 & X \arrow[dl, "\varphi"'] \arrow[dr, "\overline{\varphi}"] & \\
A & & B
\end{tikzcd}$$

  is comonadic.

- We will see that this result also holds when the above diagrams are interpreted in a setting more general than Cat.

BACKGROUND: BASIC NOTIONS

- A functor $f : A \longrightarrow B$ is bijective-on-objects if its underlying object assignment $f_0 : A_0 \longrightarrow B_0$ is a bijection.
- A functor $f : A \longrightarrow B$ is a discrete opfibration if for all pairs $(a \in A\,,\ u : fa \longrightarrow b \in B)$ there exists a unique morphism $w : a \longrightarrow a'$ in A such that $fw = u$.

Some motivating questions

1) How might we consider morphisms <u>like</u> discrete opfibrations?
2) Can we define these notions in arbitrary categories?

BACKGROUND: COFUNCTORS & LENSES

- Both cofunctors and lenses are <u>like</u> discrete opfibrations.

- A cofunctor $\varphi \colon A \nrightarrow B$ is a span of functors,

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\overline{\varphi}"] & \\ A & & B \end{tikzcd}$$

where:

φ is bijective-on-objects;

$\overline{\varphi}$ is a discrete opfibration.

- A lens $(f, \varphi) \colon A \rightleftarrows B$ is a functor $f \colon A \to B$ together with a cofunctor $\varphi \colon A \nrightarrow B$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\overline{\varphi}"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

THE SPECIFIC SETTING

Let Cat be the category of small categories and functors.

- Set is a reflective subcategory of Cat:

$$\begin{tikzcd} \mathrm{Set} \arrow[r, hook, shift right=2, "\widehat{(-)}"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, shift right=2, "(-)_0"'] \end{tikzcd}$$

underlying objects

codiscrete category

- Bijective-on-objects functors form the class of weak equivalences with respect to $(-)_0 : \mathrm{Cat} \longrightarrow \mathrm{Set}$.

- Discrete opfibrations form a chosen class of morphisms called "fibrations."

THE GENERAL SETTING

Let $\mathcal{E}$ be a category with pullbacks equipped with:

- a reflective subcategory $\mathcal{D}$ whose reflector preserves pullbacks;

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook, shift right=1.5ex, "\widehat{(-)}"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift right=1.5ex, "(-)_0"'] \end{tikzcd}$$

- the wide subcategory $\mathrm{Core}(\mathcal{E}) \subseteq \mathcal{W} \subseteq \mathcal{E}$ of weak equivalences whose members are sent to isomorphisms by the reflector;

$$\xrightarrow{\sim} \ \in \mathcal{W}$$

- a wide subcategory $\mathrm{Core}(\mathcal{E}) \subseteq \mathrm{Fib} \subseteq \mathcal{E}$ of fibrations*.

$$\twoheadrightarrow \ \in \mathrm{Fib}$$

WORKING OVER A BASE

- Let $\mathcal{E} = (\mathcal{E}, \mathcal{D}, \mathcal{W}, \mathcal{F}ib)$ be our setting for the talk.
- Choose an object $B \in \mathcal{E}$.
- Let $\mathcal{F}ib(B)$ be the full subcategory of $\mathcal{E}/B$ on the class of fibrations (think DOpf(B) in Cat).

- Let $X\mathcal{F}ib(B)$ be the category of extended fibrations over B (think Cof(B) in Cat).

objects

w.e.

fib

$$\begin{tikzcd} A \\ X \arrow[u, "\sim"'] \arrow[d, two heads] \\ B \end{tikzcd}$$

morphisms

$$\begin{tikzcd} A \arrow[rr] & & C \\ X \arrow[u, "\sim"] \arrow[rr] \arrow[dr, two heads] & & Y \arrow[u, "\sim"'] \arrow[dl, two heads] \\ & B & \end{tikzcd}$$

AN EASY LEMMA

Lemma: $\mathcal{F}ib(B)$ is a coreflective subcategory of $\mathcal{XF}ib(B)$:

$$\begin{tikzcd}
\mathcal{F}ib(B) \arrow[r, hook, shift right=2] \arrow[r, phantom, "\top"] & \mathcal{XF}ib(B) \arrow[l, shift right=2]
\end{tikzcd}$$

call the inclusion T

Proof: The counit for the adjunction is given by:

$$\begin{tikzcd}
X \arrow[rr, "\sim"] & & A \\
X \arrow[u, equal] \arrow[rr, equal] \arrow[dr] & & X \arrow[u, "\wr"'] \arrow[dl] \\
& B &
\end{tikzcd}$$

LENSES OVER A BASE

Let $\mathcal{L}ens(B)$ be the category of lenses over B consisting of:

objects

$$
\begin{tikzcd}
X \arrow[d, "s"'] \arrow[dd, bend left] \\
A \arrow[d] \\
B
\end{tikzcd}
$$

morphisms

$$
\begin{tikzcd}
X \arrow[rr] \arrow[d, "s"'] \arrow[dr] & & Y \arrow[d, "s"] \arrow[dl] \\
A \arrow[rr, dashed] \arrow[r] & B & C \arrow[l]
\end{tikzcd}
$$

Goal: Show that the forgetful functor $\mathcal{L}ens(B) \xrightarrow{L} X\mathcal{F}ib(B)$ is comonadic.

ASIDE: AN INTERESTING DIAGRAM

- Since $\mathrm{Fib}(B)$ has a terminal object 1_B, but $\mathrm{XFib}(B)$ does not, we can consider the following factorisation:

functor T_{1_B} has a right adjoint

slice of $\mathrm{XFib}(B)$ over the object $B = B = B$

$$
\begin{tikzcd}
& \mathrm{XFib}(B)/T1_B \arrow[dl, shift right=1.5] \arrow[dr, "L"] & \\
\mathrm{Fib}(B) \arrow[ur, hook, shift right=1.5, "T_{1_B}"'] \arrow[ur, phantom, "\perp"] \arrow[rr, hook, "T"'] & & \mathrm{XFib}(B)
\end{tikzcd}
$$

- Is the category $\mathrm{XFib}(B)/T1_B$ something familiar? Yes!

ASIDE: ANOTHER CHARACTERISATION OF LENSES

Proposition: There is an isomorphism $\mathcal{L}\mathrm{ens}(B) \cong \mathcal{X}\mathcal{F}\mathrm{ib}(B)/T1_B$.

Proof: Examining the objects and morphisms of $\mathcal{X}\mathcal{F}\mathrm{ib}(B)/T1_B$:

$$\begin{tikzcd}
A \arrow[rr] & & B \\
X \arrow[u, "\wr"'] \arrow[rr, two heads] \arrow[dr] & & B \arrow[u, equal] \\
& B \arrow[ur, equal] &
\end{tikzcd}$$

Objects are equivalent to lenses

$$\begin{tikzcd}
A \arrow[rr] & & B \\
X \arrow[u, "\wr"'] \arrow[rr, two heads] \arrow[dr] & & B \arrow[u, equal] \\
& B \arrow[ur, equal] &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r] & C \arrow[r] & B \\
X \arrow[u, "\wr"'] \arrow[r] \arrow[dr] & Y \arrow[u, "\wr"'] \arrow[r, two heads] \arrow[d] & B \arrow[u, equal] \\
& B \arrow[ur, equal] &
\end{tikzcd}$$

morphism in slice

Morphisms are equivalent to those in $\mathcal{L}\mathrm{ens}(B)$

UNDERLYING OBJECTS

Lemma: There is a functor $\mathrm{XFib}(B) \longrightarrow \mathcal{D}/B_0$.

Proof: On objects the functor is given by:

$$\begin{tikzcd}
A & & A_0 \arrow[d, shift left, "\varphi_0^{-1}"] & & A_0 \arrow[dd, "f_0 = \bar{\varphi}_0 \cdot \varphi_0^{-1}"] \\
X \arrow[u, "\varphi", "\wr"'] \arrow[d, "\bar{\varphi}"'] & \overset{\text{invert w.e.}}{\rightsquigarrow} & X_0 \arrow[u, shift left, "\varphi_0"] \arrow[d, "\bar{\varphi}_0"] & \overset{\text{compose}}{\rightsquigarrow} & \\
B & & B_0 & & B_0
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "(-)_0"] & \mathcal{D} \arrow[r, equal] & \mathcal{D}
\end{tikzcd}$$

CONSTRUCTING A RIGHT ADJOINT

Lemma: There is a functor $\mathrm{XFib}(B) \xrightarrow{R} \mathcal{L}\mathrm{ens}(B)$ given by:

$$
\begin{tikzcd}
 & X \arrow[dl, "\varphi"', "\sim"] \arrow[dr, "\bar{\varphi}"] & \\
A & & B
\end{tikzcd}
\longmapsto
\begin{tikzcd}
 & X \arrow[dl, "\langle \varphi, \bar{\varphi} \rangle"', "\sim"] \arrow[dr, "\bar{\varphi}"] & \\
P \arrow[rr, "\pi_B"'] & & B
\end{tikzcd}
$$

Proof: Consider the pullback:

$$
\begin{tikzcd}
X \arrow[r, dashed, "\langle \varphi, \bar{\varphi} \rangle", "\sim"'] \arrow[rr, bend left, "\bar{\varphi}"] \arrow[dr, "\varphi"', "\sim"] & P \arrow[r, "\pi_B"] \arrow[d, "\pi_A"', "\wr"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\eta_B", "\wr"'] \\
 & A \arrow[r, "\hat{f}_0 \circ \eta_A"'] & \hat{B}_0
\end{tikzcd}
$$

- η is the unit of $(-)_0 \dashv (\hat{-})$ and components are in $\mathcal{W}$ since counit $=$ id.
- $\pi_A \in \mathcal{W}$ since $(-)_0$ preserves pullbacks.
- Outer square commutes (abst. nons.)
- $\langle \varphi, \bar{\varphi} \rangle \in \mathcal{W}$ by 2-out-of-3.

THE MAIN ADJUNCTION

Lemma: There is an adjunction:

$$\begin{tikzcd} \mathcal{L}\mathrm{ens}(B) \arrow[rr, shift right=1.5ex, "L"'] \arrow[rr, phantom, "\top"] & & \mathrm{XFib}(B) \arrow[ll, shift right=1.5ex, "R"'] \end{tikzcd}$$

Proof: The unit and counit are given by:

$$\begin{tikzcd} X \arrow[rr, equal] \arrow[d, "\varphi"', "s"] \arrow[ddr, "\bar{\varphi}"] & & X \arrow[d, "s"', "\langle \varphi, \bar{\varphi} \rangle"] \arrow[ddl, "\bar{\varphi}"'] \\ A \arrow[rr, "\langle 1, f \rangle"] \arrow[dr, "f"'] & & P \arrow[dl, "\pi_B"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} P \arrow[rr, "\pi_A"] & & A \\ X \arrow[u, "\langle \varphi, \bar{\varphi} \rangle", "s"'] \arrow[rr, equal] \arrow[dr, "\bar{\varphi}"'] & & X \arrow[u, "s", "\varphi"'] \arrow[dl, "\bar{\varphi}"] \\ & B & \end{tikzcd}$$

THE MAIN RESULT

Theorem: The forgetful functor $\mathcal{L}ens(B) \xrightarrow{L} XFib(B)$ is comonadic

Proof: Consider a coalgebra for the comonad LR on $XFib(B)$:

coalgebra map

$$
\begin{tikzcd}
A \arrow[r, "\langle 1_A, f\rangle"] & P \arrow[r, "\pi_A"] \arrow[r, phantom, shift right=8ex, "\text{counit}"] & A \\
X \arrow[u, "\varphi", "s"'] \arrow[r, equal] \arrow[dr, "\bar{\varphi}"'] & X \arrow[u, "\langle \varphi, \bar{\varphi}\rangle", "s"'] \arrow[r, equal] \arrow[d, "\bar{\varphi}"] & X \arrow[u, "s", "\varphi"'] \arrow[dl, "\bar{\varphi}"] \\
& B &
\end{tikzcd}
$$

- The coalgebra maps <u>into</u> the pullback P, and compatibility with the counit gives $\langle 1_A, f\rangle : A \to P$.
- Thus a coalgebra is a functor $f : A \to B$ with $f_0 = \bar{\varphi}_0 \circ \varphi_0^{-1}$ and $f \circ \varphi = \bar{\varphi}$.
- Comultiplication adds nothing more.

$$\mathcal{L}ens(B) \simeq Coalg(LR)$$

IMMEDIATE COROLLARIES

Let $\mathcal{E} = \mathcal{C}at$, $\mathcal{D} = \mathcal{S}et$, and $\mathcal{W} = \{$bijective-on-objects functors$\}$.

- When $\mathcal{F}ib = \{$discrete opfibrations$\}$, then we have that $\mathcal{L}ens \longrightarrow \mathcal{C}of(B)$ is comonadic (this was our main goal).
- When $\mathcal{F}ib = \{$surjective-on-objects discrete opfibrations$\}$, then our coalgebras are surjective-on-objects lenses
- In general, $\mathcal{F}ib$ is any wide subcategory of $\mathcal{E}$:
  - What are some other interesting choices for $\mathcal{F}ib$?
  - How can we choose $\mathcal{F}ib$ to behave <u>like</u> discrete opfibrations ?

FIBRATIONS VIA COMONADS

Let $\mathcal{E}$ be a category with pullbacks equipped with a comonad (D, ε, δ) which preserves pullbacks.

There is a wide subcategory $\mathrm{Core}(\mathcal{E}) \subseteq$ D-$\mathcal{F}$ib $\subseteq \mathcal{E}$ of D-fibrations consisting of morphisms $f: A \to B$ in $\mathcal{E}$ such that the square

encodes the universal property

$$\begin{tikzcd} DA \arrow[r, "Df"] \arrow[d, "\varepsilon_A"'] & DB \arrow[d, "\varepsilon_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback.

Example: If $\mathcal{E} = \mathcal{C}at$ and D is the décalage comonad, then D-$\mathcal{F}$ib is the category of discrete opfibrations.

D-UNIVERSAL LENSES

A lens $(f,\varphi): A \rightleftarrows B$ in a category $\mathcal{E} = (\mathcal{E}, \mathcal{D}, \mathcal{W}, \text{D-}\mathcal{F}\text{ib})$ is **D-universal** if the morphism $Df \circ \pi_1$ is a D-fibration.

$$
\begin{tikzcd}
X \times_A DA \arrow[r, "\pi_1", "\sim"'] \arrow[d, "\pi_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[rr, bend left, "Df \circ \pi_1"] & DA \arrow[r, "Df"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\text{nat.}"] & DB \arrow[d, "\varepsilon"] \\
X \arrow[r, "\sim", "\varphi"'] \arrow[rr, bend right, "\overline{\varphi}"'] & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Example: If $\mathcal{E} = \mathcal{C}at$ and D is the décalage comonad, then a lens is D-universal if and only if it is a **split opfibration**.

REINDEXING FUNCTORS

- Given a lens $(g,\gamma): B \longrightarrow C$, there is a functor,

$$\Sigma_{(g,\gamma)} : \mathcal{L}ens(B) \longrightarrow \mathcal{L}ens(C)$$

defined by post-composition (assuming $\mathcal{F}ib$ is nice).

- Given a functor $g: B \longrightarrow C$, there is a functor,

$$\Delta_g : \mathcal{L}ens(C) \longrightarrow \mathcal{L}ens(B)$$

defined by pullback (also assuming $\mathcal{F}ib$ is nice).

- When $g \in \mathcal{F}ib$, there is an adjunction $\Sigma_g \dashv \Delta_g$.

- Do these functors yield a double category perspective?

CONNECTION TO THE ALGEBRA STORY?

- What conditions on $\mathcal{E}$ are required for the functor,

$$\mathcal{L}ens(B) \longrightarrow \mathcal{E}/B$$

to be monadic? (Recall that $\mathcal{E} = \mathcal{C}at$ is sufficient)

- What can we understand from the composite adjunction?

$$\begin{tikzcd} \mathcal{C}of(B) \arrow[r, shift right, bend right=0] \arrow[r, phantom, "\perp"] & \mathcal{L}ens(B) \arrow[l, shift right, "\text{comonadic}"'] \arrow[r, shift right, "\text{monadic}"'] \arrow[r, phantom, "\perp"] & \mathcal{C}at/B \arrow[l, shift right] \end{tikzcd}$$

- We know that $\mathcal{SO}pf(B) \longrightarrow \mathcal{C}at/B$ is monadic; is the functor $\mathcal{SO}pf(B) \longrightarrow \mathcal{C}of(B)$ also comonadic?

SUMMARY OF THE TALK

$$
\begin{tikzcd}[column sep=large, row sep=large]
\mathcal{U}\mathcal{L}ens(B) \arrow[r, hook] & \mathcal{L}ens(B) \arrow[d, phantom, "\cong"] & \\
 & XFib(B)/T1_B \arrow[dl, shift right] \arrow[dr, shift left, "L"] & \\
(D\text{-})Fib(B) \arrow[uu, hook] \arrow[ur, hook, shift right, "T1_B"'] \arrow[ur, phantom, "\top"] \arrow[rr, hook, shift right] \arrow[rr, phantom, "\top"] & & XFib(B) \arrow[ll, shift right] \arrow[ul, shift left, "R"] \arrow[ul, phantom, "\top"]
\end{tikzcd}
$$

D-universal lenses

main result

comonadic

---

preserves pullbacks

w/ pullbacks

$$
\begin{tikzcd}[column sep=huge]
\mathcal{D} \arrow[r, hook, shift right, "(\hat{-})"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift right, "(-)_0"']
\end{tikzcd}
$$

$\mathcal{W}$ - class of weak equivalences w.r.t. $(-)_0$

$\mathcal{F}ib$ - class of fibrations (possibly w.r.t. D)

D - comonad on $\mathcal{E}$ preserving pullbacks