Cofunctors, lenses, & split opfibrations

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Workshop on Polynomial Functors

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WHAT IS THIS TALK ABOUT ?

PART 1

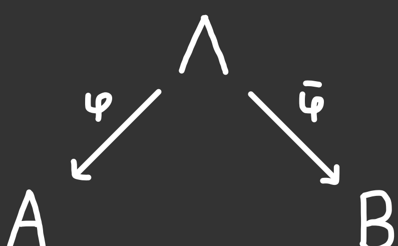
Polynomials

$$I \xleftarrow{s} E \xrightarrow{1} E \xrightarrow{t} J$$

$$\perp$$

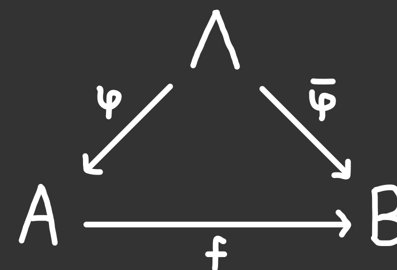
$$J \xleftarrow{t} E \xrightarrow{s} I \xrightarrow{1} I$$

Cofunctors

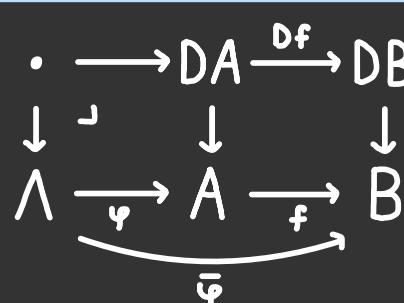


PART 2

Lenses



Split opfibrations



WHAT IS A FUNCTOR?

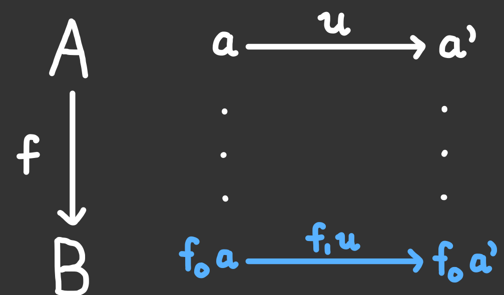
A **functor** $f : A \longrightarrow B$ between categories consists of an assignment on objects,

$$f_0 : \mathrm{Obj}(A) \longrightarrow \mathrm{Obj}(B)$$

and an assignment on morphisms,

$$f_1 : \mathrm{Mor}(A) \longrightarrow \mathrm{Mor}(B)$$

which respects domains, codomains, identities, & composition.



WHAT IS A COFUNCTOR?

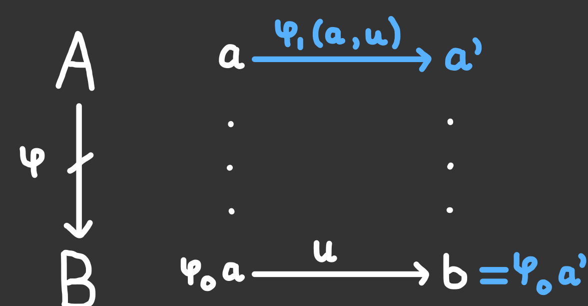
A cofunctor $\varphi : A \nrightarrow B$ between categories consists of an assignment on objects,

$$\varphi_0 : \mathrm{Obj}(A) \longrightarrow \mathrm{Obj}(B)$$

and a lifting on morphisms,

$$\varphi_1 : \mathrm{Obj}(A) \times_{\mathrm{Obj}(B)} \mathrm{Mor}(B) \longrightarrow \mathrm{Mor}(A)$$

which respects domains, codomains, identities, & composition.

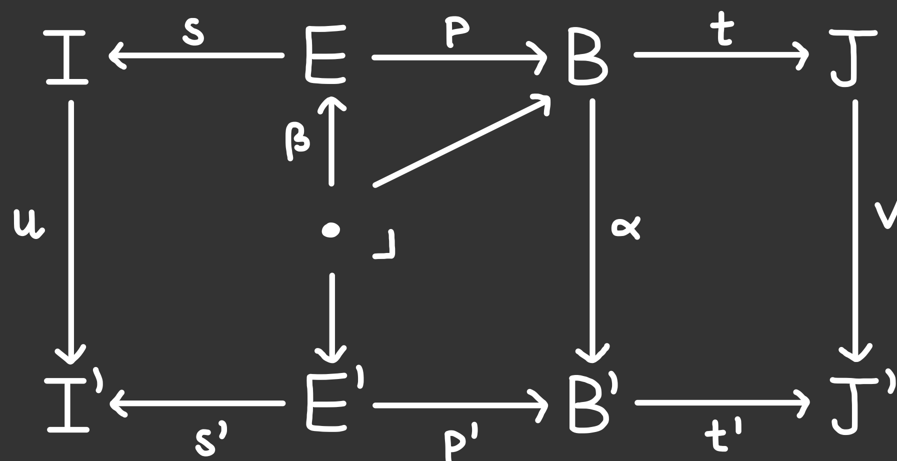


A BRIEF HISTORY OF COFUNCTORS

- 1993: Higgins & Mackenzie introduce comorphisms for vector bundles and Lie algebroids.
- 1997: Aguiar develops the notion of internal cofunctor as a dual to internal functor.
- 2016: Ahman & Uustalu prove that morphisms of polynomial comonads on Set are equivalent to cofunctors.
- 2020: Paré shows that comonad morphisms in the double category of spans and retrocells are cofunctors.

THE DOUBLE CATEGORY OF POLYNOMIALS

As shown by Gambino & Kock, for $\mathcal{E}$ with pullbacks, there is a double category $\mathbb{P}\mathrm{oly}(\mathcal{E})$ whose cells are diagrams in $\mathcal{E}$ of the form,



where the morphisms p and p' are exponentiable/powerful.

ADJOINT POLYNOMIALS

Let $\mathbb{H}\mathbb{P}\mathrm{oly}(\mathcal{E})$ be the underlying horizontal bicategory of $\mathbb{P}\mathrm{oly}(\mathcal{E})$.

Up to isomorphism, the left adjoints in $\mathbb{H}\mathbb{P}\mathrm{oly}(\mathcal{E})$ are given by,

$$I \xleftarrow{\ s\ } E \xrightarrow{\ 1\ } E \xrightarrow{\ t\ } J$$

while the corresponding right adjoints are given by:

$$J \xleftarrow{\ t\ } E \xrightarrow{\ s\ } I \xrightarrow{\ 1\ } I$$

Note that composition of left/right adjoints only requires pullbacks.

THE USUAL DOUBLE CATEGORY OF SPANS

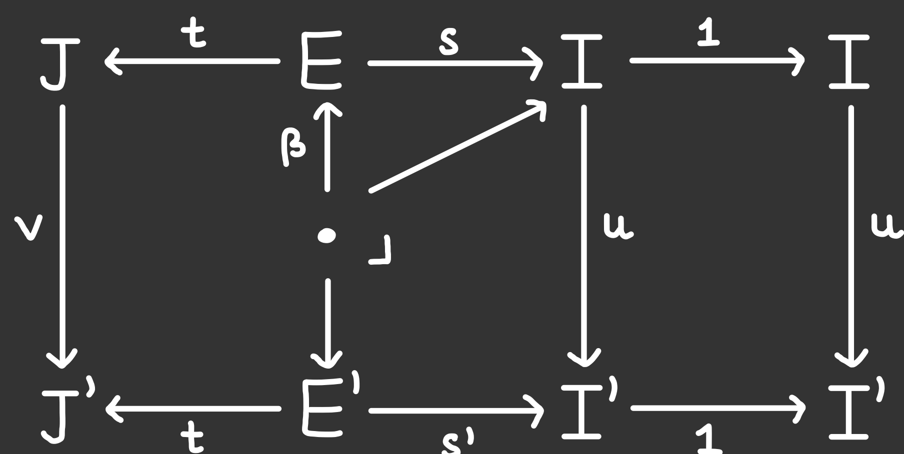
The full double subcategory of $\mathbb{P}\mathrm{oly}(\mathcal{E})$ on the left adjoints is the usual double category of spans $\mathbb{S}\mathrm{pan}(\mathcal{E})$ with cells given by:

$$\begin{array}{ccccccc} I & \xleftarrow{\;s\;} & E & \xrightarrow{\;1\;} & E & \xrightarrow{\;t\;} & J \\ {\scriptstyle u}\downarrow & & {\scriptstyle \alpha}\downarrow & \lrcorner & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle v} \\ I' & \xleftarrow[\;s'\;]{} & E' & \xrightarrow[\;1\;]{} & E' & \xrightarrow[\;t'\;]{} & J' \end{array}$$

The category of horizontal monads and vertical monad morphisms in $\mathbb{S}\mathrm{pan}(\mathcal{E})$ is equivalent to $\mathrm{Cat}(\mathcal{E})$, the category of internal categories and functors in $\mathcal{E}$.

THE DOUBLE CATEGORY OF SPANS & RETROCELLS

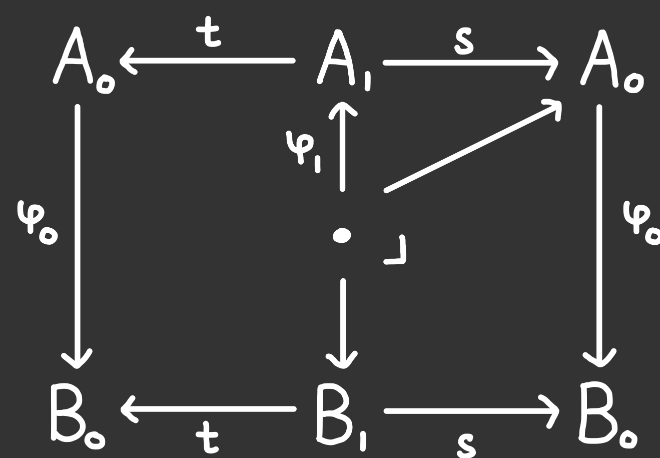
The full double subcategory of $\mathbb{P}\mathrm{oly}(\mathcal{E})$ on the right adjoints has cells given by:



This double category is equivalent to the double category of spans and retrocells $\mathbb{S}\mathrm{pan}(\mathcal{E})^{\mathrm{ret}}$ introduced by Paré.

INTERNAL COFUNCTORS

Proposition (Paré): The category of horizontal comonads and vertical comonad morphisms in $\mathcal{S}\mathrm{pan}(\mathcal{E})^{\mathrm{ret}}$ is equivalent to $\mathrm{Cof}(\mathcal{E})$, the category of internal categories and cofunctors in $\mathcal{E}$.



How is this result related to the Ahman & Uustalu characterisation?

POLYNOMIALS ON THE TERMINAL OBJECT

Suppose $\mathcal{E}$ has finite limits and let $\mathbf{Poly}_1(\mathcal{E})$ be the full double subcategory of $\mathbb{P}\mathbf{oly}(\mathcal{E})$ on the terminal object of $\mathcal{E}$.

Proposition: The inclusion $\mathbf{Poly}_1(\mathcal{E}) \hookrightarrow \mathbb{P}\mathbf{oly}(\mathcal{E})$ has a **colax left adjoint**. The counit is the identity while the unit has components given by:

$$\begin{array}{ccccccc} I & \xleftarrow{\;s\;} & E & \xrightarrow{\;p\;} & B & \xrightarrow{\;t\;} & J \\ {\scriptstyle !}\big\downarrow & & \big\downarrow{\scriptstyle 1_E} & \lrcorner & \big\downarrow{\scriptstyle 1_B} & & \big\downarrow{\scriptstyle !} \\ 1 & \xleftarrow[\;!\;]{} & E & \xrightarrow[\;p\;]{} & B & \xrightarrow[\;!\;]{} & 1 \end{array}$$

TWO VIEWS ON CATEGORIES & COFUNCTORS

There is a colax double functor given by the composite,

$$\mathbb{S}\mathrm{pan}(\mathcal{E})^{\mathrm{ret}} \xhookrightarrow{\text{pseudo}} \mathbb{P}\mathrm{oly}(\mathcal{E}) \xrightarrow{\text{colax}} \mathrm{Poly}_1(\mathcal{E})$$

which induces a functor between the categories of comonads:

$$\mathrm{Cof}(\mathcal{E}) = \mathrm{Cmd}(\mathbb{S}\mathrm{pan}(\mathcal{E})^{\mathrm{ret}}) \longrightarrow \mathrm{Cmd}(\mathrm{Poly}_1(\mathcal{E})) \qquad (*)$$

Theorem (Ahman & Uustalu): The functor (*) is an isomorphism.

This remarkable result is unintuitive and difficult to prove, but tells us something hard is actually something easy!

A DOUBLE CATEGORY OF FUNCTORS & COFUNCTORS

There is a double category $\mathbb{C}\mathrm{of}(\mathcal{E})$ of internal categories, functors, and cofunctors with flat cells given by:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \varphi\ } & B \\
{\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle g} \\
C & \xrightarrow[\ \gamma\ ]{} & D
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
A_0 & \xrightarrow{\ \varphi_0\ } & B_0 \\
{\scriptstyle f_0}\downarrow & \circlearrowright & \downarrow{\scriptstyle g_0} \\
C_0 & \xrightarrow[\ \gamma_0\ ]{} & D_0
\end{array}
\qquad
\begin{array}{ccc}
A_0 \times_{B_0} B_1 & \xrightarrow{\ \varphi_1\ } & A_1 \\
{\scriptstyle f_0 \times g_1}\downarrow & \circlearrowright & \downarrow{\scriptstyle f_1} \\
C_0 \times_{D_0} D_1 & \xrightarrow[\ \gamma_1\ ]{} & C_1
\end{array}
$$

Proposition: $\mathbb{C}\mathrm{of}(\mathcal{E})$ is span representable. Therefore $\mathbb{C}\mathrm{of}(\mathcal{E})$ has tabulators, and there is a vertically faithful double functor:

$$\mathbb{C}\mathrm{of}(\mathcal{E}) \longrightarrow \mathbb{S}\mathrm{pan}(\mathcal{C}\mathrm{at}(\mathcal{E}))$$

COFUNCTORS AS SPANS

Corollary (Higgins & Mackenzie): Every cofunctor $(\varphi_0, \varphi_1) : A \longrightarrow B$ has a faithful representation as a span of functors,

$$A \xleftarrow{\ \varphi\ } \Lambda \xrightarrow{\ \overline{\varphi}\ } B$$

where φ is bijective-on-objects and $\overline{\varphi}$ is a discrete opfibration.

Corollary: The cells of $\mathbb{C}\mathrm{of}(\mathcal{E})$ have a faithful representation as commutative diagrams of internal functors:

$$\begin{array}{ccccc} A & \xleftarrow{\ \varphi\ } & \Lambda & \xrightarrow{\ \overline{\varphi}\ } & B \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ C & \xleftarrow[\ \gamma\ ]{} & \Omega & \xrightarrow[\ \overline{\gamma}\ ]{} & D \end{array}$$

SUMMARY OF THE FIRST PART

- Functors and cofunctors appear as dual notions in $\mathbb{P}\mathrm{oly}(\mathcal{E})$.
- The category $\mathsf{Cof}(\mathcal{E})$ arises as the category of comonads and comonad morphisms in both $\mathbb{S}\mathrm{pan}(\mathcal{E})^{\mathrm{ret}}$ and $\mathrm{Poly}_1(\mathcal{E})$.
- There is a double category $\mathbb{C}\mathrm{of}(\mathcal{E})$ whose cells are diagrams:

$$
\begin{array}{ccccc}
A & \xleftarrow{\ \varphi\ } & \Lambda & \xrightarrow{\ \bar{\varphi}\ } & B \\
{\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\
C & \xleftarrow[\ \gamma\ ]{} & \Omega & \xrightarrow[\ \bar{\gamma}\ ]{} & D
\end{array}
$$

φ, γ bijective-on-objects

$\bar{\varphi}, \bar{\gamma}$ discrete opfibration

WHAT IS A LENS?

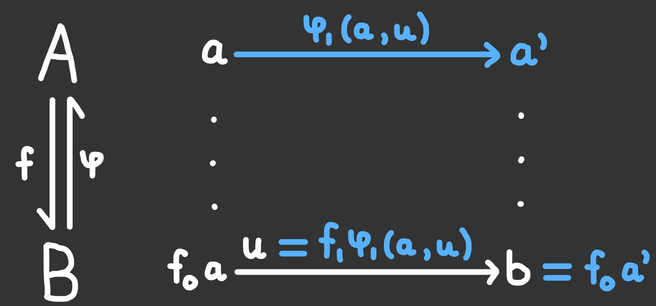
A (delta) lens $(f, \varphi) : A \rightleftarrows B$ between categories consists of assignments on objects and morphisms,

$$f_0 : \mathrm{Obj}(A) \longrightarrow \mathrm{Obj}(B) \qquad\qquad f_1 : \mathrm{Mor}(A) \longrightarrow \mathrm{Mor}(B)$$

and a lifting on morphisms,

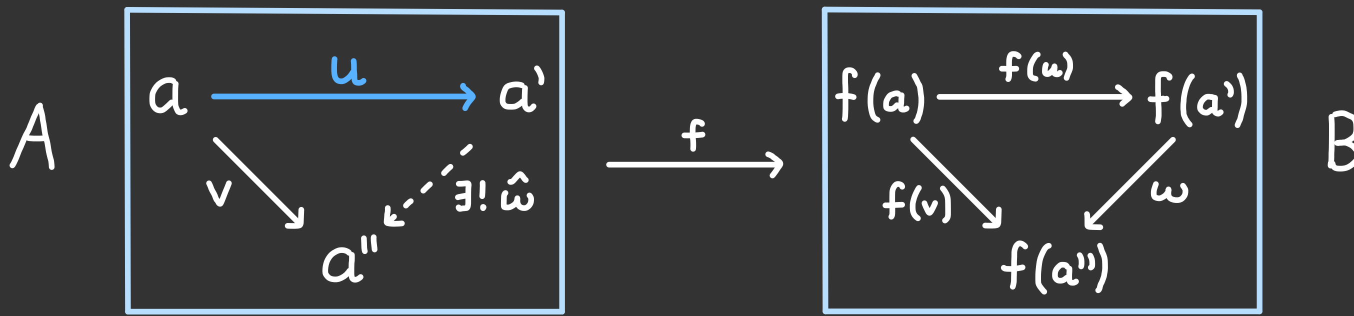
$$\varphi_1 : \mathrm{Obj}(A) \times_{\mathrm{Obj}(B)} \mathrm{Mor}(B) \longrightarrow \mathrm{Mor}(A)$$

which respect domains, codomains, identities, & composition.



WHAT IS A SPLIT OPFIBRATION?

- A morphism $u : a \longrightarrow a'$ in A is opcartesian with respect to a functor $f : A \longrightarrow B$ if for all $v : a \longrightarrow a''$ in A and for all $w : fa' \longrightarrow fa''$ in B such that $w \circ f(u) = f(v)$, there exists a unique $\hat{w} : a' \longrightarrow a''$ in A such that $\hat{w} \circ u = v$ and $f(\hat{w}) = w$.



- A split opfibration is a lens whose chosen lifts are opcartesian.

A BRIEF HISTORY OF LENSES

- 2005: Foster, Greenwald, Moore, Pierce, & Schmitt introduce lenses between sets ($g: A \longrightarrow B$, $p: A \times B \longrightarrow A$) for computer science.
- 2011: Diskin, Czarnecki, & Xiong develop the notion of delta lens between categories.
- 2013: Johnson & Rosebrugh prove that every split opfibration is a lens.
- 2017: Ahman & Uustalu show that lenses may be understood in terms of compatible functor and cofunctor pairs.

A BRIEF HISTORY OF SPLIT OPFIBRATIONS

- 1966: Gray reviews Grothendieck fibrations and introduces several equivalent characterisations.
- 1974: Street develops the theory of fibrations in a 2-category and characterises split opfibrations as algebras for a monad.
- 1977: Johnstone defines internal split opfibrations as internal categories in $\mathrm{DOpf}(\mathcal{E})/B$.
- 2017: Ahman & Uustalu show that split opfibrations can be defined as lenses with additional structure.

LENSES VIA FUNCTORS & COFUNCTORS

Proposition (Ahman & Uustalu): A lens $(f, \varphi) : A \rightleftarrows B$ is equivalent to a functor $f : A \to B$ and a cofunctor $\varphi : A \nrightarrow B$ such that $f_0 = \varphi_0$ and

$$\mathrm{Obj}(A) \times_{\mathrm{Obj}(B)} \mathrm{Mor}(B) \xrightarrow{\ \varphi_1\ } \mathrm{Mor}(A) \xrightarrow{\ f_1\ } \mathrm{Mor}(B)$$

$$\mathrm{Obj}(A) \times_{\mathrm{Obj}(B)} \mathrm{Mor}(B) \xrightarrow{\ \mathrm{proj}\ } \mathrm{Mor}(B)$$

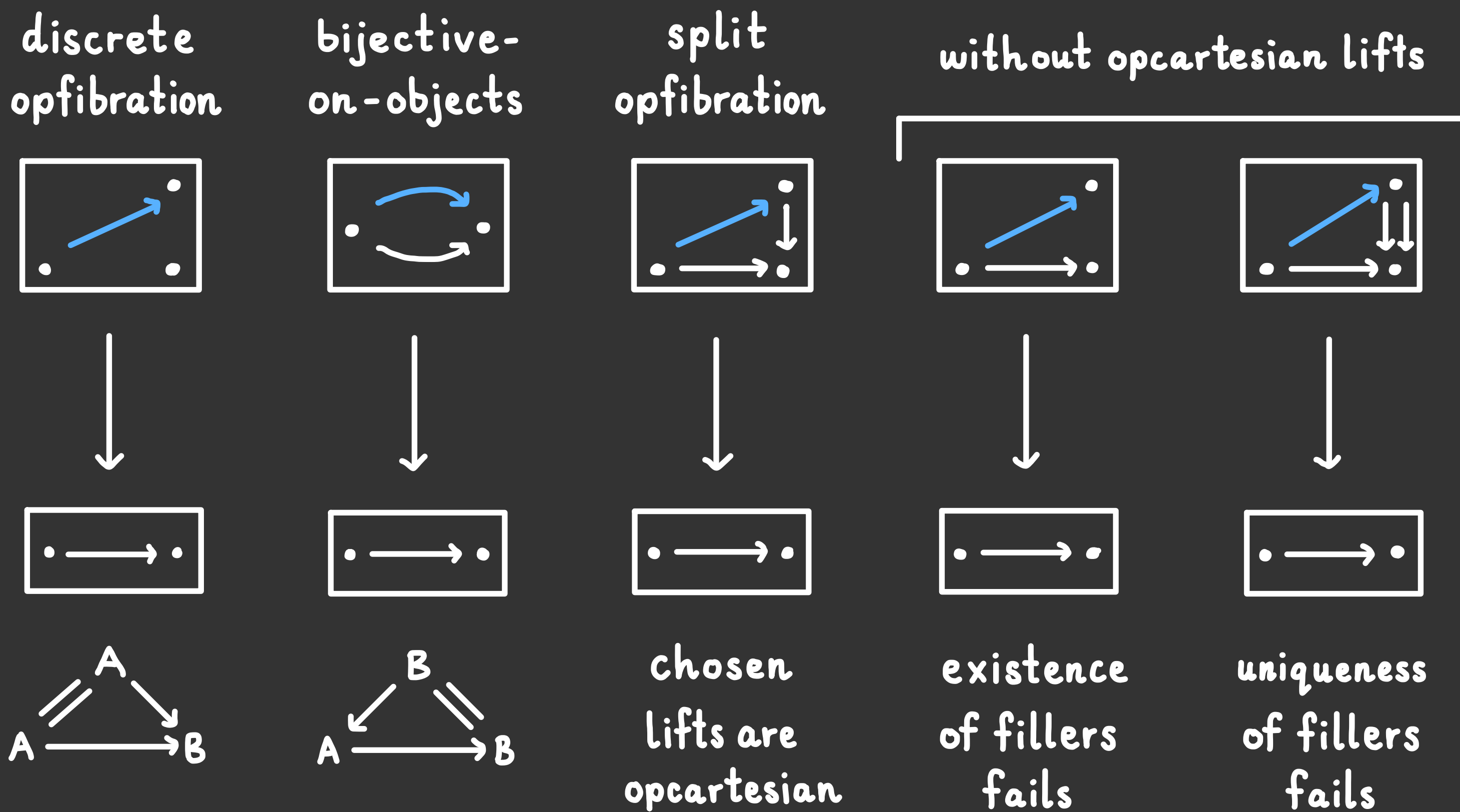
Corollary: Every lens $(f, \varphi) : A \rightleftarrows B$ has a faithful representation as a diagram of functors:

$$\begin{array}{ccccc} & & \Lambda & & \\ & \varphi \swarrow & & \searrow \overline{\varphi} & \\ A & & \xrightarrow{\quad f \quad} & & B \end{array}$$

φ bijective-on-objects

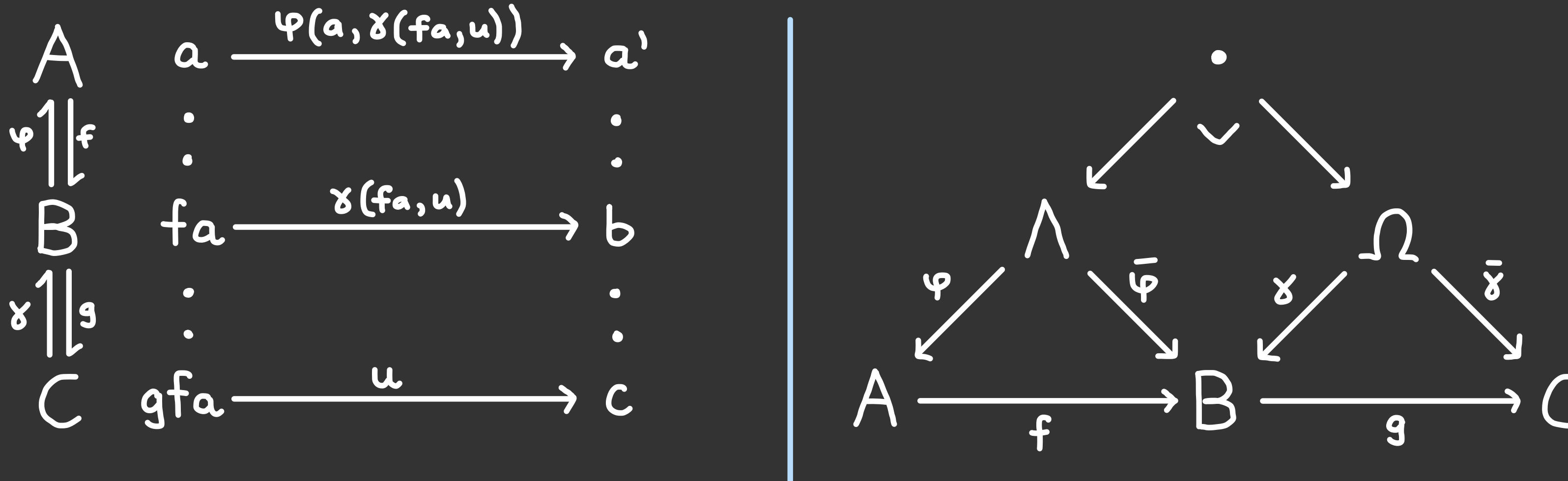
$\overline{\varphi}$ discrete opfibration

BASIC EXAMPLES OF LENSES



THE CATEGORY OF LENSES

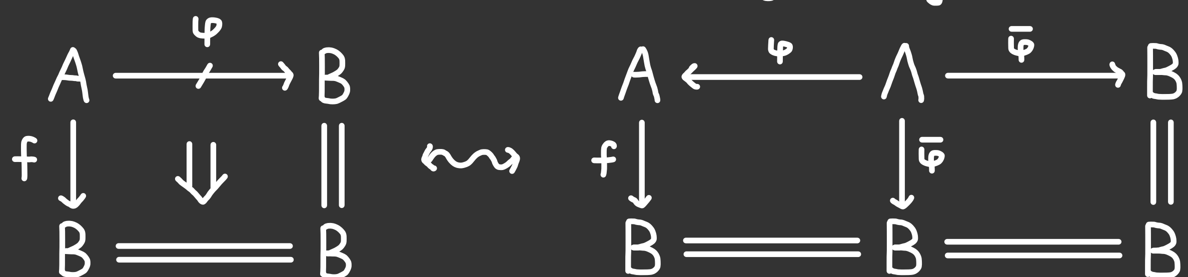
There is a category $\mathcal{L}ens$ whose objects are categories and whose morphisms are lenses, with composition given by:



How can we view lenses arising from cofunctors? Internalise in $\mathcal{E}$?

THE DOUBLE CATEGORY OF LENSES

Given the double category $\mathbb{C}\mathrm{of}(\mathcal{E})$, we may construct a double category with the same objects and vertical morphisms as $\mathbb{C}\mathrm{of}(\mathcal{E})$, with horizontal morphisms $A \nrightarrow B$ given by cells of the form,



and with cells given by cells in $\mathbb{C}\mathrm{of}(\mathcal{E})$ satisfying a pasting law.

Proposition: This construction yields the double category $\mathbb{L}\mathrm{ens}(\mathcal{E})$ of internal categories, functors, and lenses. Let $\mathcal{L}\mathrm{ens}(\mathcal{E}) := H\mathbb{L}\mathrm{ens}(\mathcal{E})$.

THE DÉCALAGE CONSTRUCTION

- Given a category A, the **décalage** of A is the sum of its slice categories:

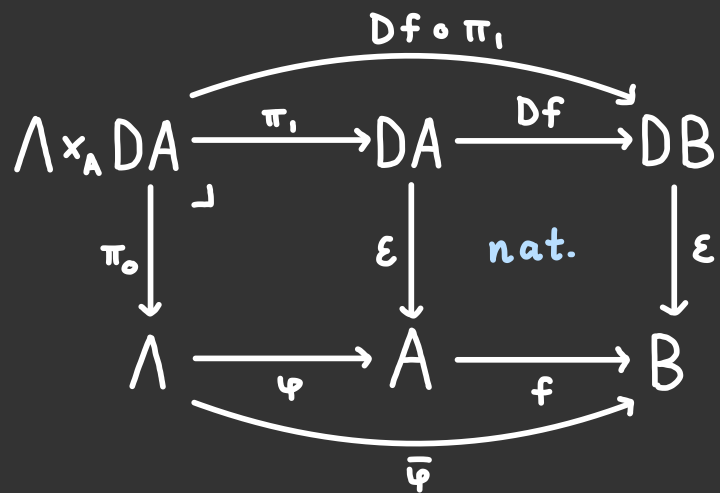
$$\mathrm{Dec}(A) = \sum_{a \in A} A/a$$

- Décalage generalises to a **comonad** $D : \mathrm{Cat}(\mathcal{E}) \longrightarrow \mathrm{Cat}(\mathcal{E})$ whose counit is a discrete fibration:

$$\begin{array}{c} DA \\ \varepsilon \downarrow \\ A \end{array} \qquad\qquad \begin{array}{ccccc} A_1 & \xleftarrow{\;c\;} & A_2 & \longrightarrow & A_1 \\ {\scriptstyle s}\downarrow & & \downarrow \lrcorner & & \downarrow{\scriptstyle s} \\ A_0 & \xleftarrow[\;s\;]{} & A_1 & \xrightarrow[\;t\;]{} & A_0 \end{array}$$

SPLIT OPFIBRATIONS VIA DÉCALAGE

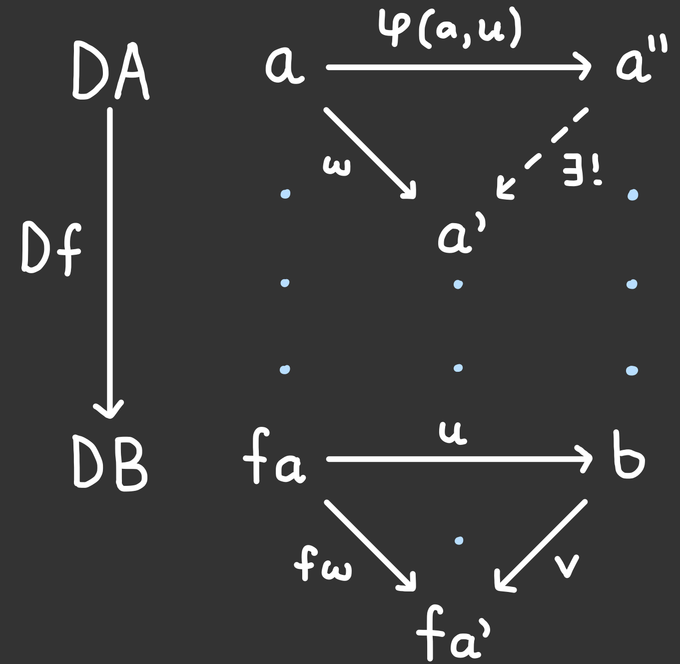
Theorem: A lens $(f, \varphi) : A \rightleftarrows B$ between internal categories is a split opfibration if and only if the functor $Df \circ \pi_1$ given by,



is a discrete opfibration, where D is the décalage comonad.

CONSEQUENCES & FUTURE WORK

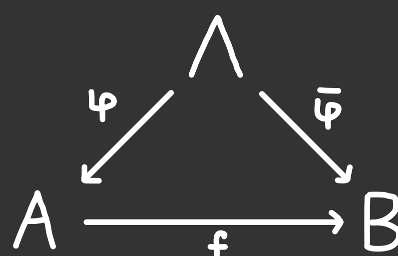
- When $(f,\varphi): A \rightleftarrows B$ is a split opfibration, the functor Df has a lens structure.
- The characterisation is compact, and directly generalises the $\mathcal{E} = Set$ case.
- Suggests a way of defining split opfibrations internally without using 2-categories



SUMMARY OF THE TALK

- Functors and cofunctors arise dually within the double category $\mathbb{P}\mathrm{oly}(\mathcal{E})$ of polynomials.

- Lenses are morphisms between categories which are both functors and cofunctors in a compatible way.



φ bijective-on-objects

$\overline{\varphi}$ discrete opfibration

- Split opfibrations are lenses which satisfy a property with respect to décalage.

LIST OF REFERENCES

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- Aguiar (1997). Internal categories and quantum groups.
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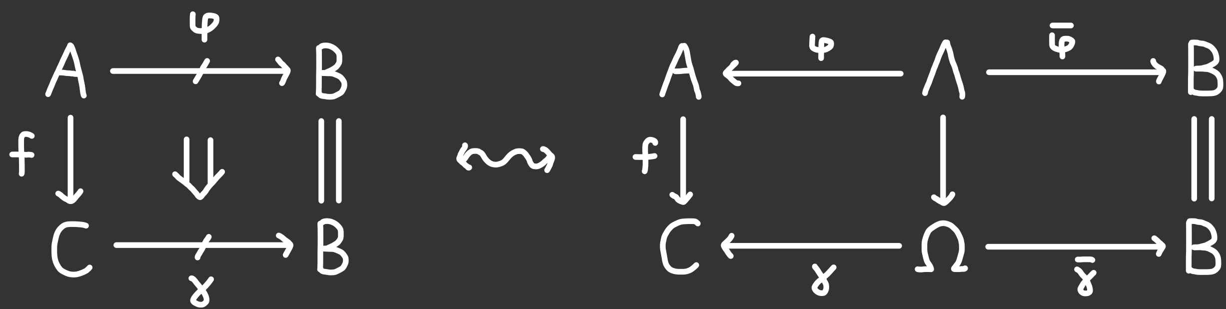
- Foster, Greenwald, Moore, Pierce, & Schmitt (2005). Combinators for bidirectional tree transformations: A linguistic approach to the view-update problem.
- Diskin, Czarnecki, & Xiong (2011). From state- to delta-based bidirectional model transformations: the asymmetric case.
- Johnson & Rosebrugh (2013). Delta lenses and opfibrations.
- Ahman & Uustalu (2017). Taking updates seriously.

LIST OF REFERENCES

- Gray (1966). Fibred and cofibred categories.
- Street (1974). Fibrations and Yoneda's lemma in a 2-category.
- Johnstone (1977). Topos Theory.

BONUS. COFUNCTORS OVER A BASE

In the double category $\mathbb{C}\mathrm{of}(\mathcal{E})$ we may consider cells of the form:



(*)

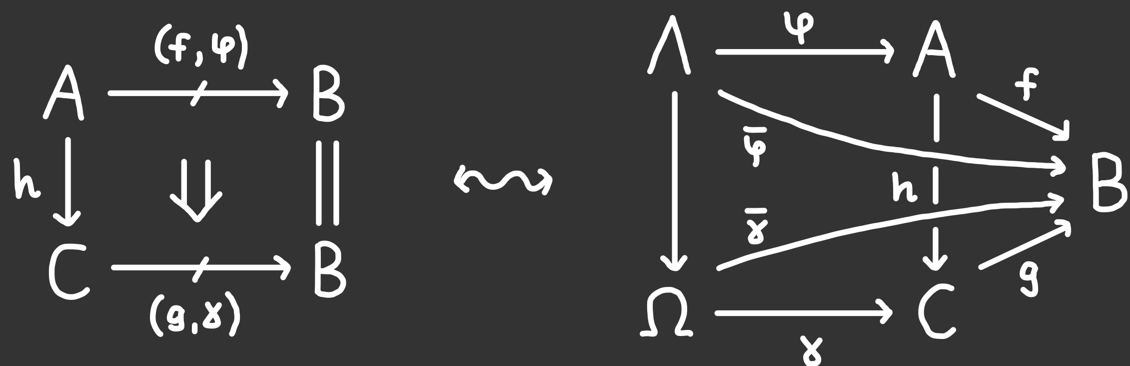
Then for each internal category B, let $\mathrm{Cof}_B(\mathcal{E})$ denote the category of cofunctors over a base B whose:

- objects are cofunctors with codomain B;
- whose morphisms are cells in $\mathbb{C}\mathrm{of}(\mathcal{E})$ of the form (*).

BONUS: LENSES OVER A BASE

For each internal category B, let $\mathcal{L}\mathrm{ens}_B(\mathcal{E})$ denote the category of lenses over a base B whose:

- objects are lenses with codomain B;
- whose morphisms are cells in $\mathbb{L}\mathrm{ens}(\mathcal{E})$ of the form:

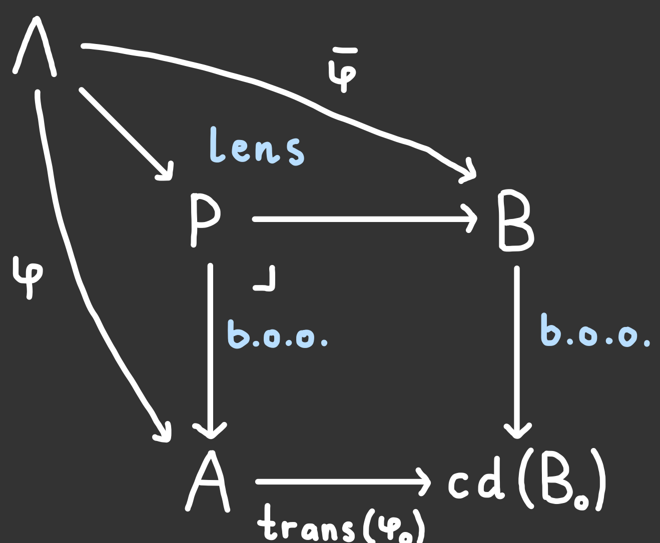


Proposition: There is an isomorphism $\mathcal{L}\mathrm{ens}_B(\mathcal{E}) \cong \mathcal{C}\mathrm{of}_B(\mathcal{E}) / 1_B$

BONUS: LENSES AS COALGEBRAS FOR A COMONAD

Theorem: The functor $\mathcal{L}ens_B(\mathcal{E}) \xrightarrow{\mathcal{U}} \mathcal{C}of_B(\mathcal{E})$ is **comonadic**.

Proof (sketch): Given a cofunctor $\varphi : A \nrightarrow B$, there is a lens $P \rightleftarrows B$.



This defines a right adjoint $R \dashv \mathcal{U}$, and the **coalgebras** for the comonad $R\mathcal{U}$ are: