The Grothendieck Construction for Lenses

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The Calgary Peripatetic Seminar

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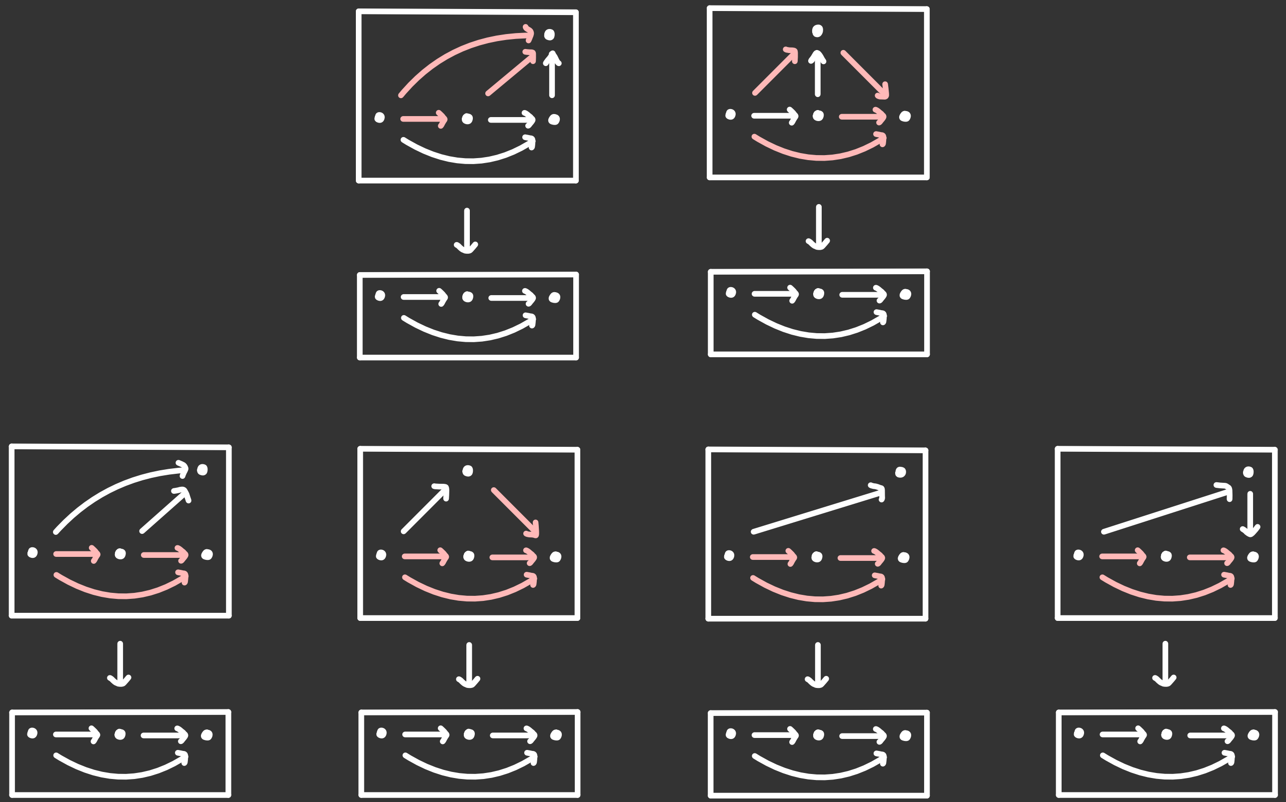
MOTIVATION: SPLIT OPFIBRATIONS

MOTIVATION: LENSES



OUTLINE OF THE TALK

THE CATEGORY OF ELEMENTS

Given a functor $F : B \longrightarrow \mathrm{Set}$, we can construct the comma category:

$$\begin{array}{ccc} \int F & \longrightarrow & 1 \\ {\scriptstyle \pi}\downarrow & \swArrow & \downarrow{\scriptstyle *} \\ B & \underset{F}{\longrightarrow} & \mathrm{Set} \end{array}$$

The category of elements $\int F$ has:

- objects $(b \in B,\ x \in F_b)$
- morphisms $(b, x) \longrightarrow (b', x')$ where $u : b \longrightarrow b'$ such that $x' = F_u(x)$.

The projection functor,

$$\begin{array}{ccc} \int F & \longrightarrow & B \\ (b, x) & \longmapsto & b \end{array}$$

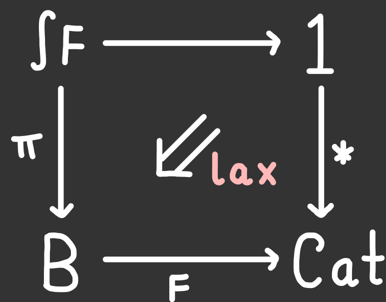
is a discrete opfibration.

There is an equivalence of categories:

$$\mathrm{DOpf}(B) \simeq [B, \mathrm{Set}]$$

THE GROTHENDIECK CONSTRUCTION

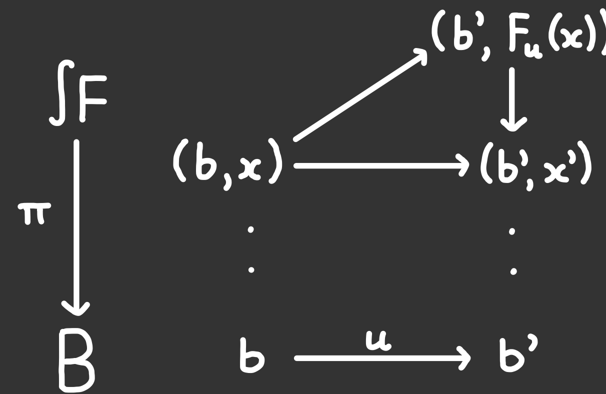
Given a functor $F : B \longrightarrow \mathrm{Cat}$, we can construct the comma category:



The category $\int F$ has:

- objects $(b \in B,\ x \in F_b)$
- morphisms $(b, x) \longrightarrow (b', x')$ are $u : b \to b'$ and $\alpha : F_u(x) \longrightarrow x'$.

The projection functor,



is a split opfibration.

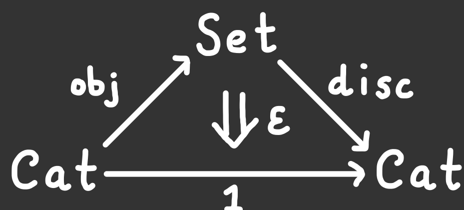
There is an equivalence of categories:

$$\mathrm{SOpf}(B) \simeq [B, \mathrm{Cat}]$$

SPLITTING FROM AN ADJUNCTION

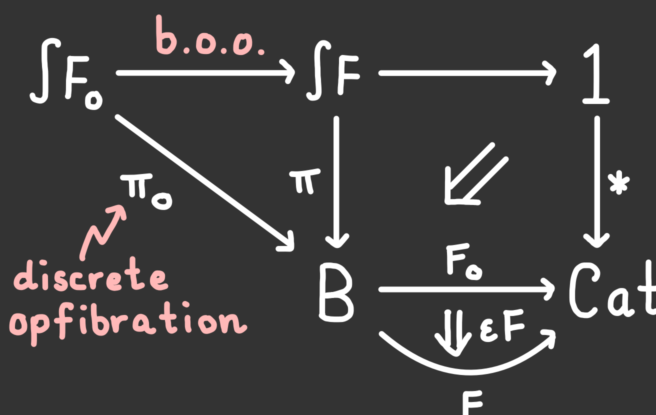
- A split opfibration is an opfibration with the additional structure of a splitting. How do we obtain this structure?

- First recall the adjunction disc ⊣ obj with counit,



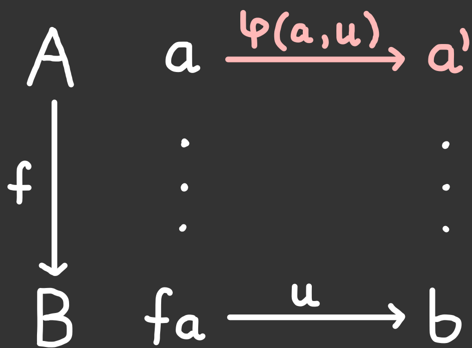
and whisker with $F: B \longrightarrow Cat$.

- Then use the universal property of the comma:



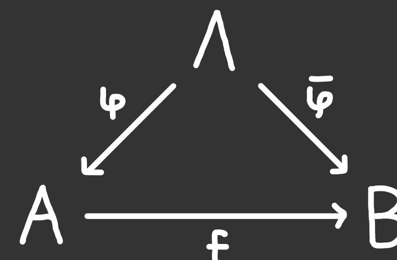
INTRODUCING (DELTA) LENSES

- A **lens** is a functor equipped with a suitable choice of lifts.



- A **split opfibration** is a lens with a certain property: the chosen lifts $\varphi(a,u)$ are **opcartesian**.

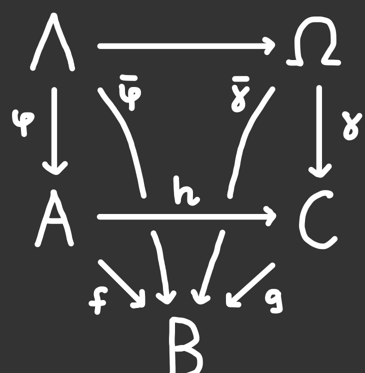
Proposition: Every lens $A \xrightarrow{(f,\varphi)} B$ has a faithful representation as a commutative diagram of functors,



where φ is **bijective-on-objects** and $\bar{\varphi}$ is a **discrete opfibration**.

A GROTHENDIECK CONSTRUCTION FOR LENSES?

- For each small category B, there is a category $\mathcal{L}ens(B)$ whose:
  - objects are lenses into B;
  - morphisms are given by:



i.e. functors $h : A \longrightarrow C$ such that $gh = f$ & $h\varphi(a,u) = \gamma(ha,u)$.

- There are full subcategories:

$$DOpf(B) \hookrightarrow \mathcal{L}ens(B) \quad (\text{with a dashed arrow } T \text{ back})$$

$$SOpf(B) \hookrightarrow \mathcal{L}ens(B)$$

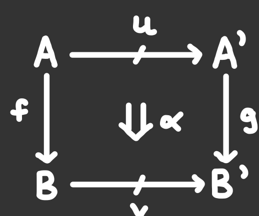
- Central question: is there a (double) category $\mathbb{D}$ such that there is an equivalence of categories,

$$\mathcal{L}ens(B) \simeq [B, \mathbb{D}] \ ?$$

DOUBLE CATEGORIES

A **double category** $\mathbb{D}$ consists of:

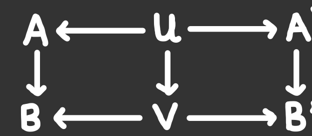
- objects $A, B, \ldots$
- vertical morphisms $f: A \to B, \ldots$
- horizontal morphisms $u: A \nrightarrow A', \ldots$
- cells



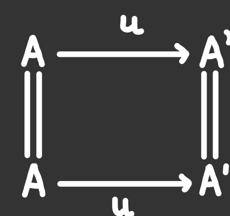
Horizontal composition is associative up to comparison isocells.

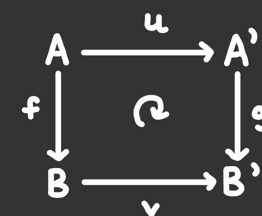
Examples:

- $\mathbb{S}\mathrm{pan}$ - sets, functions, spans



- For each category B, we have $\mathbb{H}B$ and $\mathbb{Q}B$ with cells (resp.):



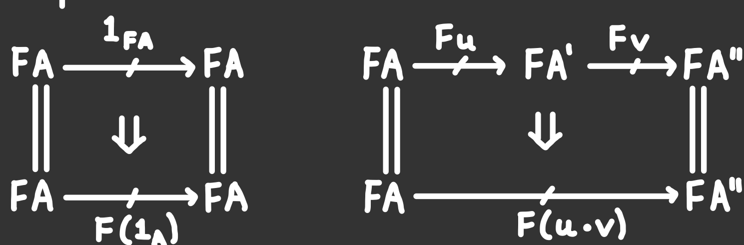


- The terminal double cat. $\mathbb{1}$

LAX DOUBLE FUNCTORS & TRANSFORMATIONS

- A lax double functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ consists of an assignment,

which preserves vertical direction strictly & horizontal direction up to coherence cells:



- A vertical transformation $t : F \Rightarrow G : \mathbb{A} \longrightarrow \mathbb{B}$ consists of an assignment,

which satisfies naturality and coherence conditions.

- For each pair of double categories $\mathbb{A}$ and $\mathbb{B}$, there is a category $[\mathbb{A}, \mathbb{B}]_{lax}$.

FUNCTORS AS LAX DOUBLE FUNCTORS

Given a lax double functor $F : \mathbb{H}B \longrightarrow \mathbb{S}\mathrm{pan}$, we can construct the comma category:

$$\begin{array}{ccc} \int F & \longrightarrow & 1 \\ {\scriptstyle \pi}\big\downarrow & \swarrow_{\text{lax}} & \big\downarrow{\scriptstyle *} \\ \mathbb{H}B & \xrightarrow[F]{} & \mathbb{S}\mathrm{pan} \end{array}$$

The category $\int F$ has morphisms:

$$\left(b \xrightarrow{u} b' \, , \quad \begin{array}{ccccc} & & 1 & & \\ & {\scriptstyle x}\swarrow & \downarrow{\scriptstyle \alpha} & \searrow{\scriptstyle x'} & \\ F_b & \longleftarrow & F_u & \longrightarrow & F_{b'} \end{array} \right)$$

- The laxity of F induces identities and composition in $\int F$.
- The projection $\pi : \int F \longrightarrow \mathbb{H}B \simeq$ a functor $H(\int F) \xrightarrow{\pi} B$ given by $\pi(u, \alpha) = u : b \longrightarrow b$.
- There is an equivalence of categories:

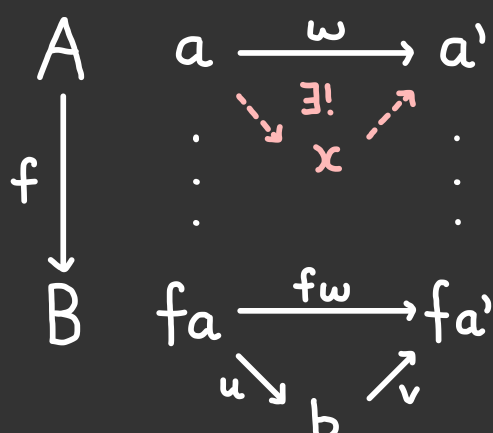
$$\mathrm{Cat}/B \simeq [\mathbb{H}B, \mathbb{S}\mathrm{pan}]_{\text{lax}}$$

(see Paré "Yoneda theory..." for details)

SPECIAL KINDS OF FUNCTORS

- $\mathbb{HB} \longrightarrow \mathbb{S}\mathrm{pan}$ which are normal
  $\simeq$ functors with discrete fibres.

- $\mathbb{HB} \longrightarrow \mathbb{S}\mathrm{pan}$ which are pseudo
  $\simeq$ discrete Conduche fibrations



- $\mathbb{HB} \longrightarrow \mathbb{Q}\mathrm{Set} \longrightarrow \mathbb{S}\mathrm{pan}$
  $\simeq$ discrete opfibrations

- $\mathbb{HB} \longrightarrow \mathbb{R}\mathrm{el} \longrightarrow \mathbb{S}\mathrm{pan}$
  $\simeq$ faithful functors

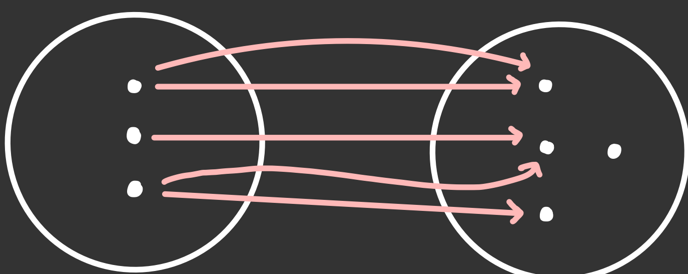
- $\mathbb{HB} \longrightarrow \mathbb{P}\mathrm{ar} \longrightarrow \mathbb{S}\mathrm{pan}$
  $\simeq$ faithful functors $f: A \longrightarrow B$ s.t.
  if $f(a \xrightarrow{u} a') = f(a \xrightarrow{v} a'')$,
  then $u = v$ (uniqueness of lifts).

MULTI-VALUED FUNCTIONS

Let $\mathbb{M}\text{ult}$ be the full double subcategory of $\mathbb{S}\text{pan}$ on spans of the form.

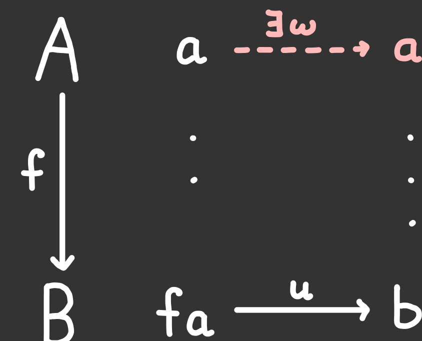
$$X \overset{\text{epi}}{\twoheadleftarrow} Z \longrightarrow Y$$

These are called **multi-valued functions** (not necessarily relations).



- $\mathbb{HB} \longrightarrow \mathbb{M}\text{ult} \longrightarrow \mathbb{S}\text{pan}$

$\simeq$ functors with **existence of lifts**:

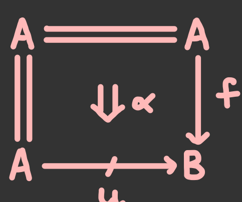


- This is a **necessary condition** for a functor to have a lens structure.

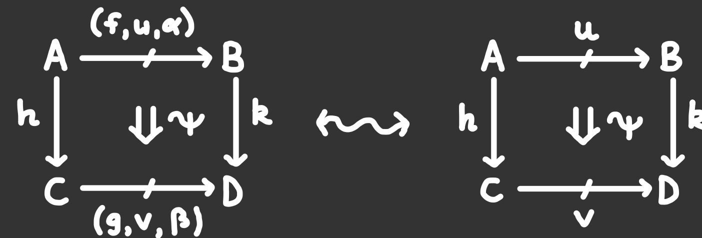
A CONSTRUCTION ON DOUBLE CATEGORIES

Given a double category $\mathbb{D}$, we may construct a double category $\tilde{\mathbb{D}}$ whose:

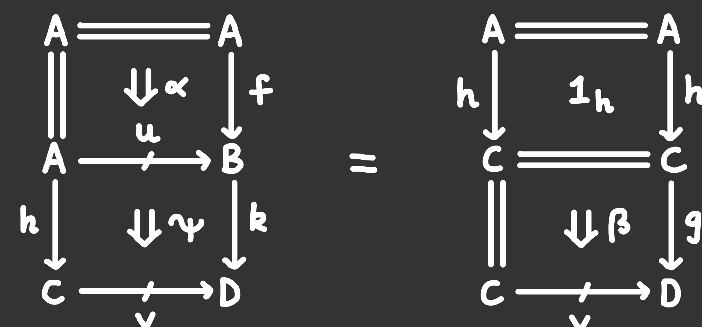
- objects & vertical morphisms are the same as $\mathbb{D}$;
- horizontal morphisms are given by cells in $\mathbb{D}$ of the form:



- cells are given by cells in $\mathbb{D}$,

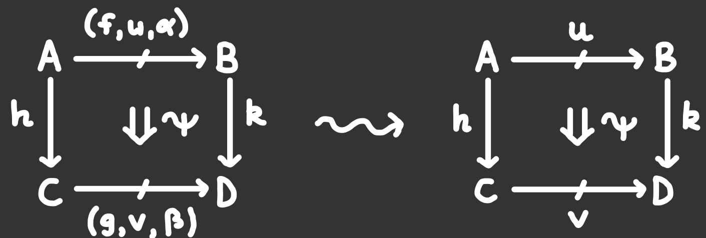


such that:



CONNECTION TO COMPANIONS

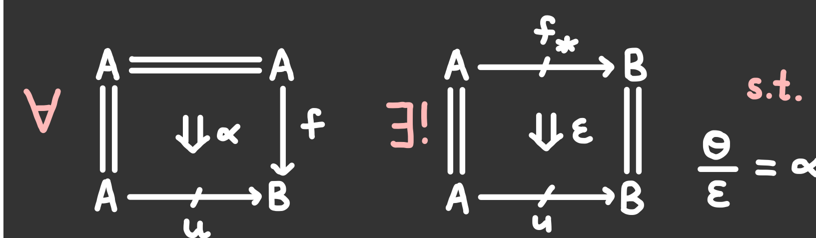
- There is a pseudo double functor $\widetilde{\mathbb{D}} \longrightarrow \mathbb{D}$ given by the assignment



- There is also a pseudo double functor $\widetilde{\mathbb{D}} \xrightarrow{\mathcal{U}} \mathbb{Q}(V\mathbb{D})$ given by:

- If $\mathbb{D}$ has companions, $\mathcal{U}$ has a left adjoint given by:

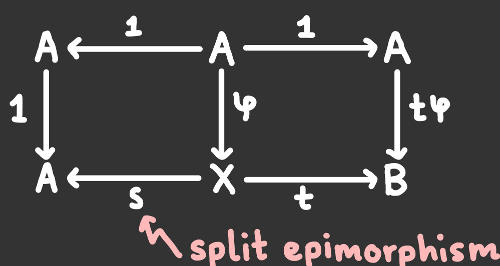
- The counit is given by the universal property of the companion cell:



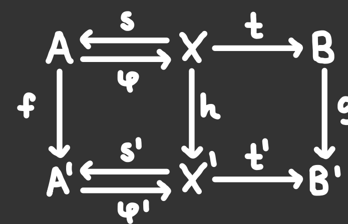
SPLIT MULTI-VALUED FUNCTIONS

- We may apply our construction to $\mathbb{D} = \mathbb{S}\mathrm{pan}$ to obtain $\widetilde{\mathbb{D}} = \mathrm{s}\mathbb{M}\mathrm{ult}$, the double category of split multi-valued functions.

- A horizontal morphism in $\mathrm{s}\mathbb{M}\mathrm{ult}$ is a cell in $\mathbb{S}\mathrm{pan}$ of the form:

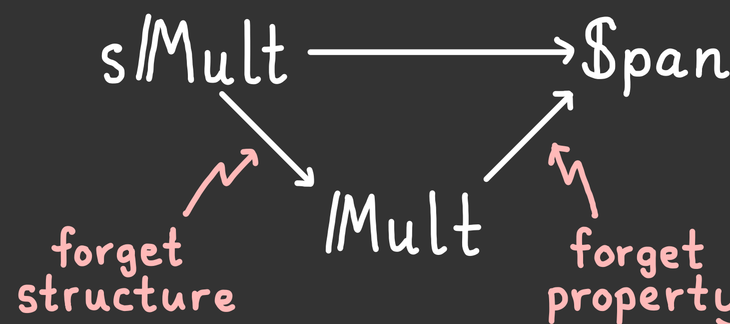


- A cell in $\mathrm{s}\mathbb{M}\mathrm{ult}$ is given by:



$$gt = t'h$$
$$fs = s'h$$
$$h\varphi = \varphi' f$$

- The forgetful functor factors:

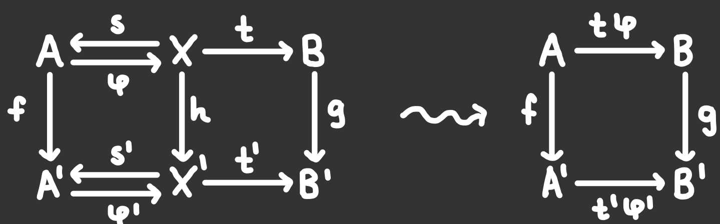


AN ADJUNCTION OF DOUBLE CATEGORIES

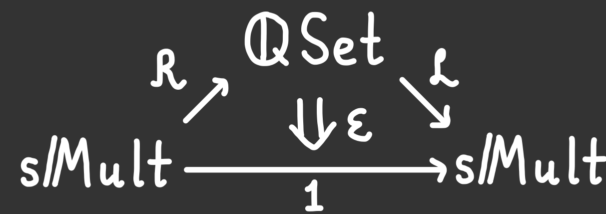
- There is an adjunction of double categories:

$$\mathbb{Q}\mathrm{Set} \underset{\mathcal{R}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathrm{s/Mult} \qquad (\mathcal{L} \dashv \mathcal{R})$$

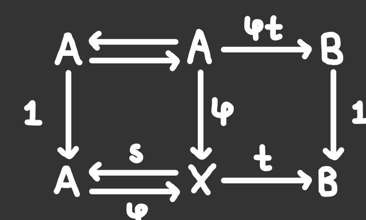
- The right adjoint is given by:



- The counit of the adjunction,



has components which take a split multi-valued function to the cell.

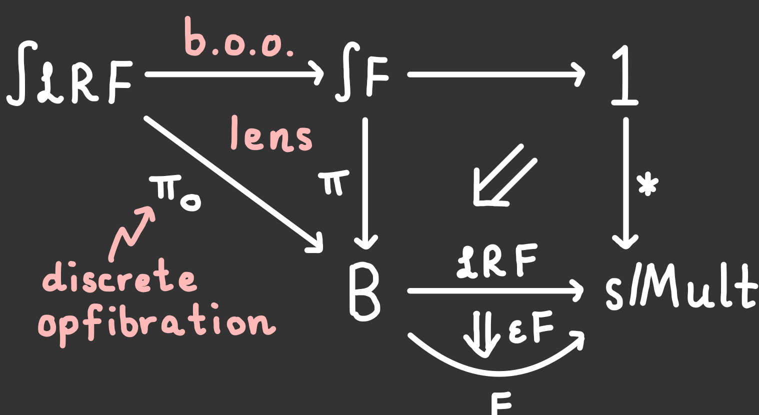


MAIN THEOREM

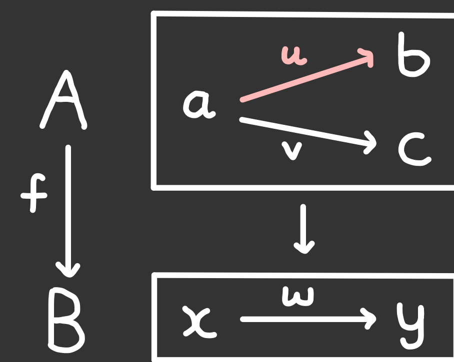
Theorem: There is an equivalence of categories:

$$\mathcal{L}ens(B) \simeq [\mathbb{IH}B, s/Mult]_{lax}$$

Proof (idea): Lax functor $\rightsquigarrow$ lens

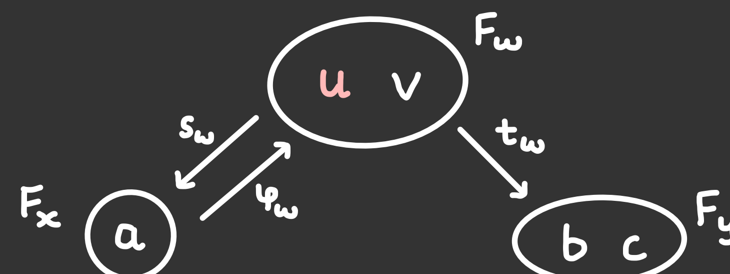


Conversely, consider a lens:

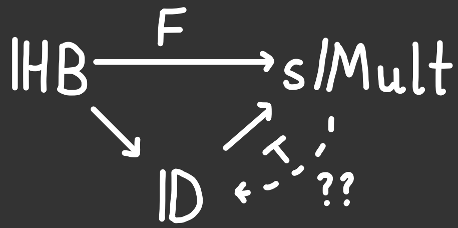


where $\varphi(a,w) = u$.

This gives a $\mathbb{IH}B \longrightarrow s/Mult$:



SPECIAL KINDS OF LENSES



- Discrete opfibration $\simeq$ $\mathbb{D} = \mathbb{Q}\mathbf{Set}$

- Cosieves $\simeq$ $\mathbb{D} = \mathbb{Q}\{0 \to 1\}$

- Bijective-on-objects lens $\simeq$

  $\mathbb{D}$ is full double subcategory on

$$1 \underset{p}{\overset{!}{\rightleftarrows}} X \xrightarrow{!} 1$$

- Fully faithful lens $\simeq$

  $\mathbb{D}$ is full double subcategory on

$$A \underset{\langle 1, f\rangle}{\overset{\pi_0}{\rightleftarrows}} A \times B \xrightarrow{\pi_1} B$$

- Discrete fibration* $\simeq$

  $\mathbb{D}$ is full double subcategory on

$$1_C \; A \underset{\sigma}{\overset{p}{\rightleftarrows}} B \xrightarrow{1} B$$

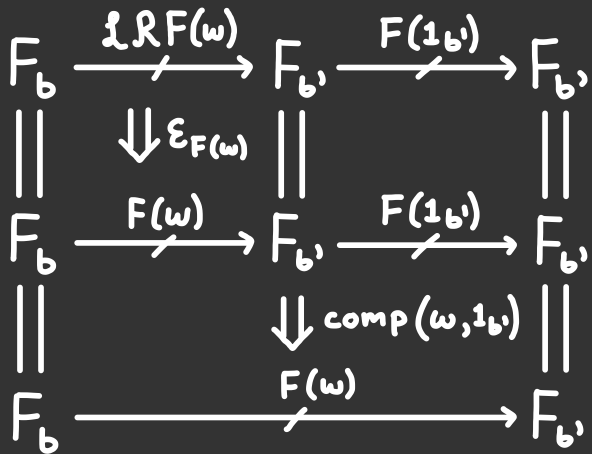
*chosen section to each function between fibres

SPLIT OPFIBRATIONS REVISITED

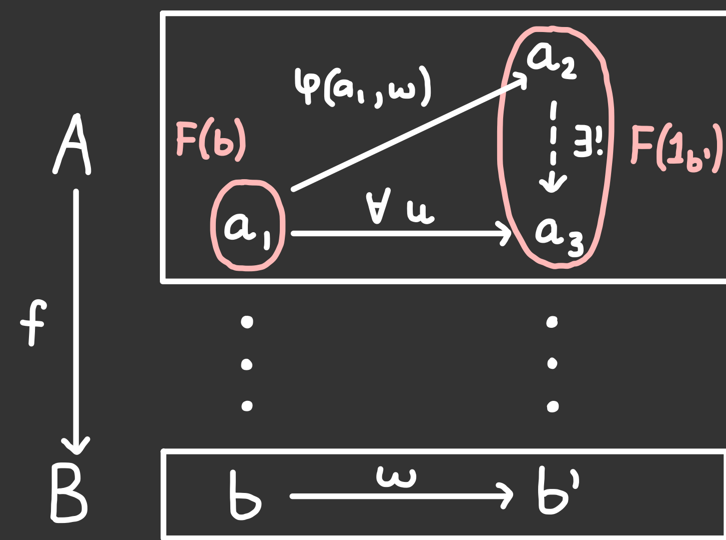
Split opfibration $\simeq$

Lax double functor $F : \mathbb{H}\mathcal{B} \to \mathsf{s/Mult}$

such that the cell,



is an isocell for all $w : b \to b' \in \mathcal{B}$.



SUMMARY AND FUTURE WORK

- We discussed the Grothendieck construction at several levels.

$$\mathsf{DOpf}(B) \simeq [B, \mathsf{Set}]$$
$$\mathsf{SOpf}(B) \simeq [B, \mathsf{Cat}]$$
$$\mathsf{Cat}/B \simeq [\mathbb{H}B, \mathbb{S}\mathsf{pan}]_{lax}$$

- Introduced a construction $\widetilde{\mathbb{D}}$ such that when $\mathbb{D} = \mathbb{S}\mathsf{pan}$, $\widetilde{\mathbb{D}} = \mathsf{s}\mathbb{M}\mathsf{ult}$.
- Established a Grothendieck construction for lenses:

$$\mathcal{L}\mathsf{ens}(B) \simeq [\mathbb{H}B, \mathsf{s}\mathbb{M}\mathsf{ult}]_{lax}$$

- What is theory underlying the construction $\widetilde{\mathbb{D}}$, and are there more examples?
- The double functor $\mathsf{s}\mathbb{M}\mathsf{ult} \to \mathbb{S}\mathsf{pan}$ induces a monadic functor:

$$[\mathbb{H}B, \mathsf{s}\mathbb{M}\mathsf{ult}]_{lax} \longrightarrow [\mathbb{H}B, \mathbb{S}\mathsf{pan}]_{lax}$$

Explicit description of left adjoint?
- Can we generalise further for categories B with more structure?