What's so nice about the category of lenses?

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COLLABORATORS

This talk is based on research from the Applied Category Theory 2020 Adjoint School together with:

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The goal of the project (and this talk) was to investigate the "nice" properties of the category $\mathcal{L}$**ens** of categories and (delta) lenses.

OUTLINE OF THE TALK

Background	- What is a lens? - Some basic examples
Motivation	- The problem with pullbacks - A partial solution via discrete opfibrations
Results	- Products, equalisers, coproducts, extensivity - Factorisation systems, epis, monos
Conclusion	- Ideas for future work - Summary of the talk

REVIEWING LENSES

A **lens** $(f, \varphi) : A \rightleftarrows B$ is a functor equipped with a **choice of lifts**,

$$\begin{tikzcd} A \arrow[d, shift left, "f"] & a \arrow[r, "\varphi(a,u)"] \arrow[d, phantom, "\vdots"] & a' \arrow[d, phantom, "\vdots"] \\ B \arrow[u, shift left, "\varphi"] & fa \arrow[r, "u"] & b \end{tikzcd}$$

which satisfies the axioms:

- $\varphi(a, 1_{fa}) = 1_a$
- $\varphi(a, v \circ u) = \varphi(a', v) \circ \varphi(a, u)$

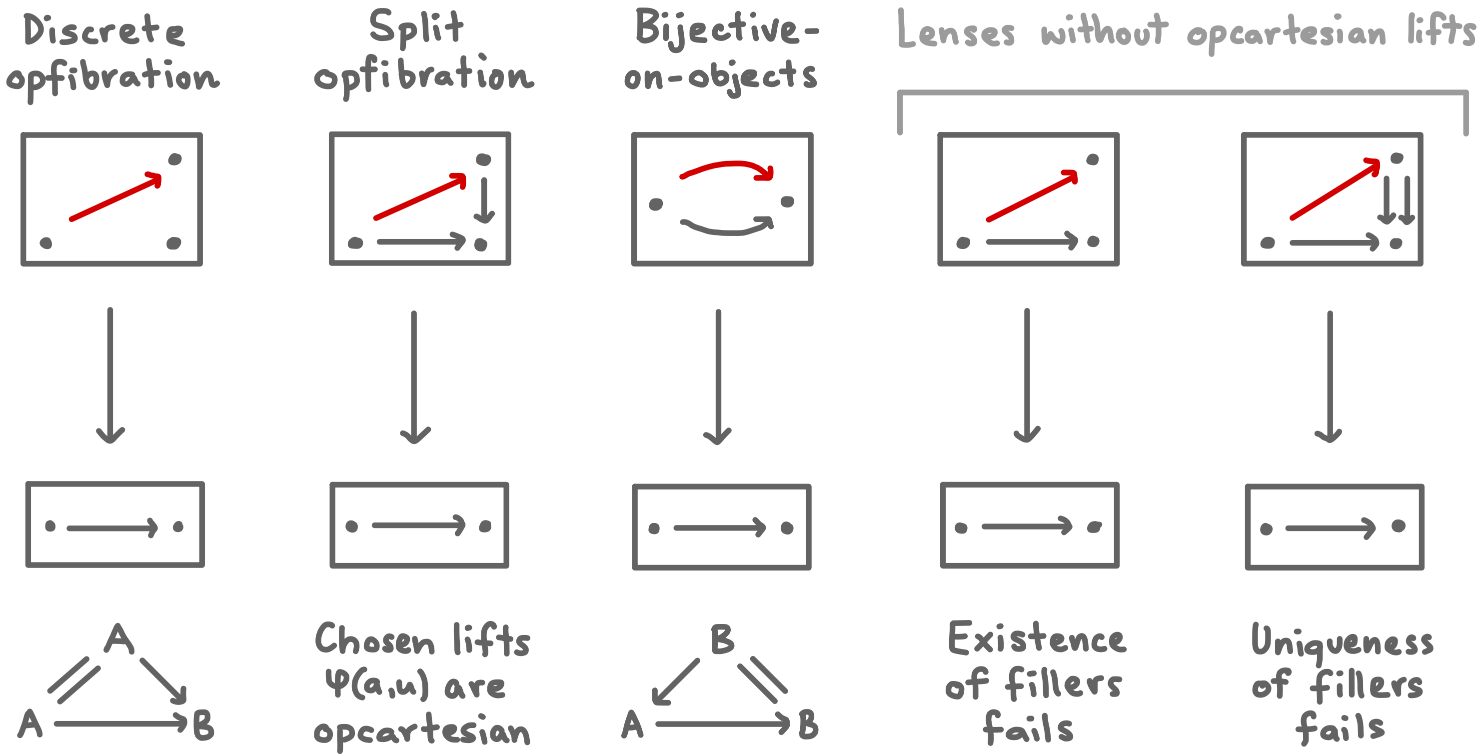
- Every lens $(f, \varphi) : A \rightleftarrows B$ may be represented by a commutative diagram in $\mathcal{C}at$,

$$\begin{tikzcd} & \Lambda \arrow[dl, "\varphi"'] \arrow[dr, "\overline{\varphi}"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

where φ is **bijective-on-objects** and $\overline{\varphi}$ is a **discrete opfibration.**

- The span $(\overline{\varphi}, \Lambda, \varphi) : B \nrightarrow A$ is called a **cofunctor.**

BASIC EXAMPLES OF LENSES



THE CATEGORY OF LENSES

There is a category **Lens** whose objects are categories and whose morphisms are lenses with composition given by:

$$
\begin{tikzcd}
A \arrow[d, shift right, "f"] & a \arrow[rrr, "\varphi(a,\gamma(fa,u))"] \arrow[d, dotted, no head] & & & a' \arrow[d, dotted, no head] \\
B \arrow[u, shift right, "\varphi"] \arrow[d, shift right, "g"] & fa \arrow[rrr, "\gamma(fa,u)"] \arrow[d, dotted, no head] & & & b \arrow[d, dotted, no head] \\
C \arrow[u, shift right, "\gamma"] & gfa \arrow[rrr, "u"] & & & c
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \bullet \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & \\
& \Lambda \arrow[dl, "\varphi"'] \arrow[dr, "\bar{\varphi}"] & & \Omega \arrow[dl, "\gamma"'] \arrow[dr, "\bar{\gamma}"] & \\
A \arrow[rr, "f"'] & & B \arrow[rr, "g"'] & & C
\end{tikzcd}
$$

There is an identity-on-objects isofibration $\mathcal{L}ens \longrightarrow \mathcal{C}at$ which assigns a lens to its underlying functor.

THE PROBLEM WITH PULLBACKS

- A category $\mathcal{E}$ has pullbacks $\Leftrightarrow$ $\mathcal{E}/B$ has products for all $B \in \mathcal{E}$.
- The category $\mathbf{Lens}/B$ has a **monoidal product** given by:

$$
\begin{tikzcd}
A \times_B \Omega \arrow[rr, "1 \times \gamma"] \arrow[dr, "\bar{\pi}_A"'] & & A \times_B C \arrow[dl, "\pi_A"] \arrow[dr, "\pi_C"'] \arrow[d, phantom, "\lrcorner"] & & \Lambda \times_B C \arrow[ll, "\varphi \times 1"'] \arrow[dl, "\bar{\pi}_C"] \\
& A \arrow[dr, "f"] & & C \arrow[dl, "g"'] & \\
\Lambda \arrow[ur, "\varphi"] \arrow[rr, "\bar{\varphi}"'] & & B & & \Omega \arrow[ul, "\gamma"'] \arrow[ll, "\bar{\gamma}"]
\end{tikzcd}
$$

- **Problem**: this monoidal structure is not cartesian in general!
- Is it possible that the universal property holds for certain inputs?

DIGRESSION: SOME USEFUL LEMMAS

Consider a diagram in $\mathcal{C}at$ with g a discrete opfibration:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g \circ f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd} \qquad (*)$$

Then:

(1) $g \circ f$ is a discrete opfibration $\Longrightarrow$ f is a discrete opfibration

(2) $g \circ f$ has a lens structure $\Longrightarrow$ f has a unique lens structure such that $(*)$ commutes in $\mathcal{L}ens$

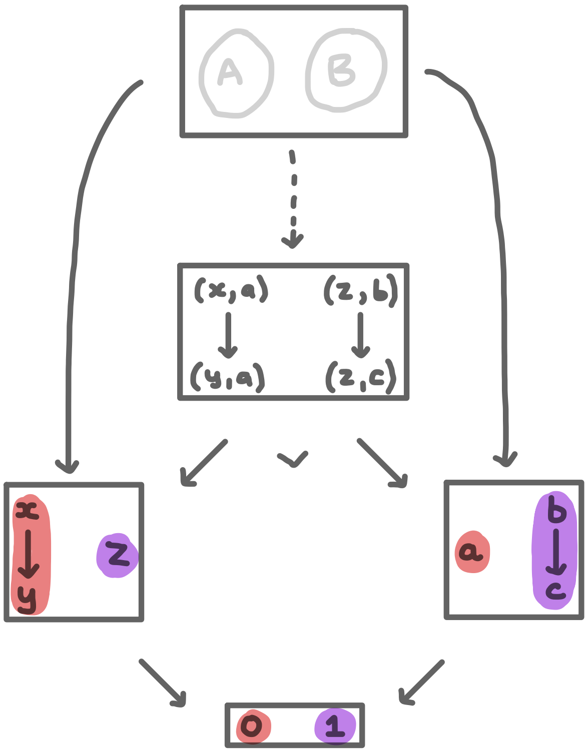
PULLBACKS ALONG DISCRETE OPFIBRATIONS

Proposition: The category $\mathcal{L}\mathrm{ens}$ admits all pullbacks along discrete opfibrations. Moreover, these pullbacks coincide with the "canonical" monoidal structure on $\mathcal{L}\mathrm{ens}/B$.

Proof: Apply the previous lemma to the triangle (*) to give the universal functor $Z \to A \times_B C$ a lens structure.

$$
\begin{tikzcd}
 & & & Y \arrow[dll] \arrow[dddr, bend left] & \\
 & Z \arrow[ddrrr, bend left] \arrow[ddr, dashed, "(*)"'] \arrow[dddr, bend right] & & & \\
 & & & \Lambda \times_B C \arrow[dl] \arrow[dr] & \\
X \arrow[uur, bend left] \arrow[drr, bend right] & & A \times_B C \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & C \arrow[d] \\
 & & A \arrow[rr] & & B \\
 & & & \Lambda \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

OTHER EXAMPLES OF PULLBACKS



Any other commuting square of lenses into the cospan is equivalent to a pair of lenses from the connected components as follows:

Thus the universal property is satisfied.

Conjecture: If all products of fibres involve a discrete cat, the pullback exists.

MONOIDAL STRUCTURE & PRODUCTS

The cartesian monoidal structure on Cat induces a **semi-cartesian monoidal structure** on $\mathcal{L}$ens with projections given by:

$$\begin{tikzcd}
& A \times B_0 \arrow[dl, "\overline{\pi}_A"'] \arrow[dr, "1 \times i"] & & A_0 \times B \arrow[dl, "i \times 1"'] \arrow[dr, "\overline{\pi}_B"] & \\
A & & A \times B \arrow[ll, "\pi_A"] \arrow[rr, "\pi_B"'] & & B
\end{tikzcd}$$

Moreover, if A or B is a **discrete category**, then the above corresponds to the **cartesian product** in $\mathcal{L}$ens.

Open question: Are these the only products in $\mathcal{L}$ens?

COPRODUCTS

Proposition: The canonical functor $\mathcal{L}ens \longrightarrow \mathcal{C}at$ **creates coproducts.**

Proof (idea): Recall that in $\mathcal{C}at$, the coproduct injection functors are injective-on-objects discrete opfibrations.

Given a pair of lenses $(f, \varphi) : A \longrightarrow B$ and $(g, \gamma) : C \longrightarrow B$ we have a unique lens from the coproduct in $\mathcal{C}at$ commuting with the injections given by:

$$\begin{tikzcd}
& \Lambda + \Omega \arrow[dl, "\varphi + \gamma"'] \arrow[dr, "{[\bar{\varphi}, \bar{\gamma}]}"] & \\
A + C \arrow[rr, "{[f, g]}"'] & & B
\end{tikzcd}$$

DISTRIBUTIVITY

- A monoidal category is **distributive** if the canonical map,

$$A \otimes B + A \otimes C \xrightarrow{[1 \otimes i_B,\, 1 \otimes i_C]} A \otimes (B + C)$$

  is an isomorphism for all A, B, C.

- Since $\mathcal{C}at$ distributive (with respect to the cartesian monoidal structure), and $\mathcal{L}ens \longrightarrow \mathcal{C}at$ is a strong monoidal isofibration, we have that **$\mathcal{L}ens$ is distributive**.*

- Actually we can show that distributivity follows from an even stronger property of $\mathcal{L}ens$.

EXTENSIVITY

- A category with coproducts is **extensive** if pullbacks along coproduct injections exist, and in any commutative diagram,

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & Z \arrow[d] & Y \arrow[l] \arrow[d] \\
A \arrow[r, "\iota_A"'] & A+B & B \arrow[l, "\iota_B"]
\end{tikzcd}
$$

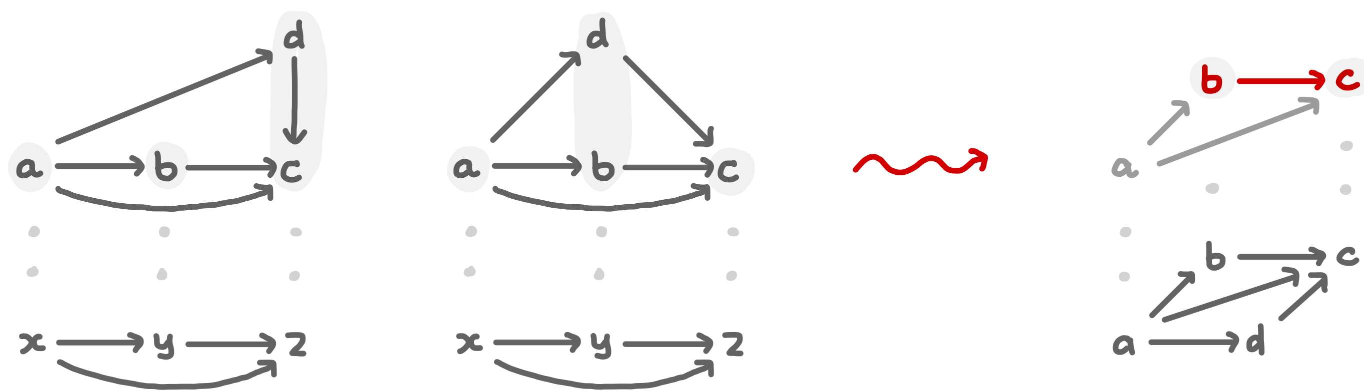
the two squares are pullbacks $\Leftrightarrow$ the top row is a coproduct.

- Proposition: $\mathcal{L}$ens is an extensive category.

EQUALISERS

- **Lens admits all equalisers**, however unlike pullbacks and products, they are not always preserved by Lens → Cat.

- We may construct equalisers in Lens through taking the **largest subobject*** of the equaliser in Cat which admits a lens structure and equalises the diagram in Lens.



MONOMORPHISMS

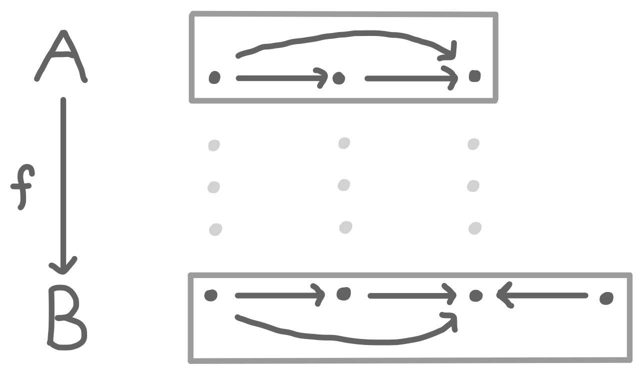
A lens $(f,\varphi)\colon A \rightleftarrows B$ is a **monomorphism** if any of the following equivalent conditions hold:

(1) (f,φ) is an **injective-on-objects discrete opfibration**;

(2) (f,φ) is a **fully faithful discrete opfibration**;

(3) f is a **monomorphism** in $\mathcal{C}at$.

Conjecture: These conditions are necessary and sufficient.



EPIMORPHISMS

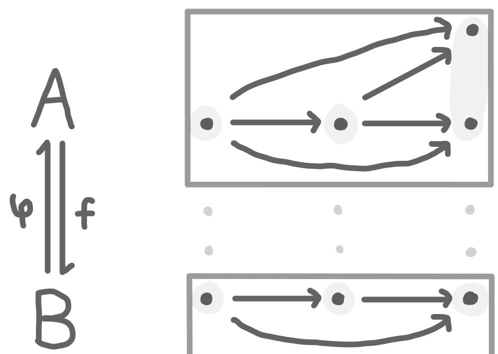
A lens $(f,\varphi): A \rightleftarrows B$ is an **epimorphism** if any of the following equivalent conditions hold:

(1) (f,φ) is **surjective-on-objects** ;

(2) (f,φ) is **surjective-on-morphisms** .

Conjecture: These conditions are necessary and sufficient.

Corollary (?): In $\mathcal{L}ens$, epi + mono $\iff$ isomorphism.



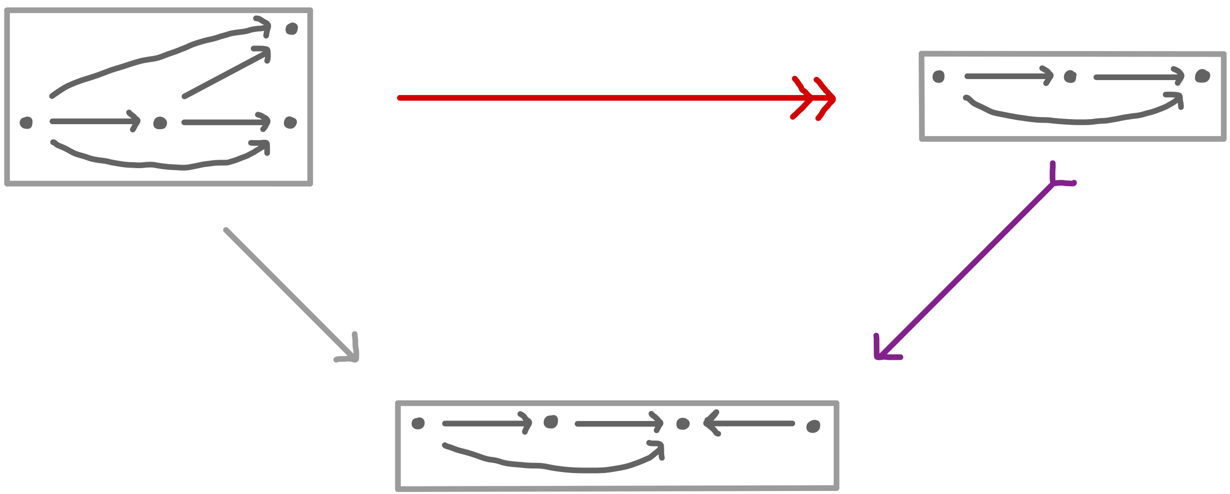
PROPER FACTORISATION SYSTEM

The category $\mathcal{L}ens$ has an orthogonal factorisation system with:

$\mathcal{E}$ = surjective-on-objects lenses $\subseteq \mathrm{Epi}(\mathcal{L}ens)$

$\mathcal{M}$ = injective-on-objects discrete opfibrations $\subseteq \mathrm{Mono}(\mathcal{L}ens)$

This corresponds to the (surj-on-ob, inj-on-ob f.f.) OFS via $\mathcal{L}ens \to \mathcal{C}at$.



SPLIT IDEMPOTENTS

- An idempotent splits if the following parallel pair has an equaliser (or coequaliser):

$$\begin{tikzcd} Q \arrow[r, dashed] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "1_A"'] & A \arrow[r, dashed] & Q \end{tikzcd}$$

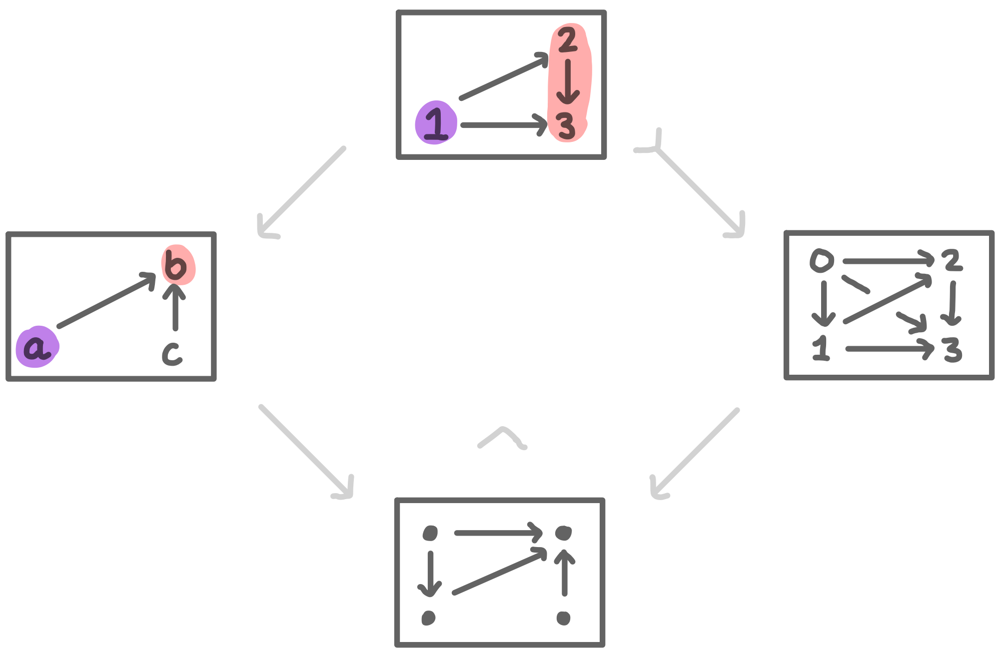
- In $\mathcal{L}$ens, all idempotents split and they may be obtained by the (epi, mono) factorisation:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] & & A \\ & Q \arrow[ur, tail, "m"'] & \end{tikzcd}$$

- These give simple examples of coequalisers — are there others?

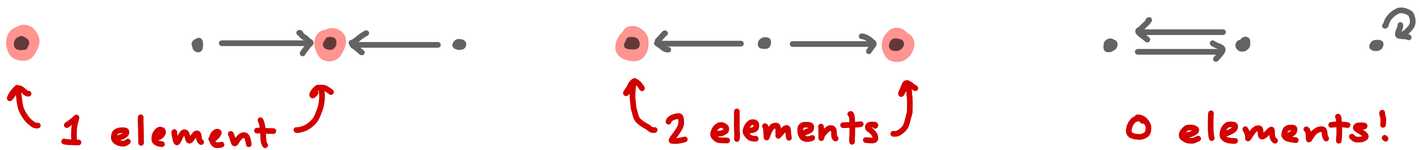
PUSHOUTS & ADHESIVITY ?

- Like coequalisers, it seems difficult to construct pushouts in $\mathcal{L}$ens, but do we at least have them along monos ?



ELEMENTS & SINKS

- For categories and functors we have $\mathcal{C}at(1, A) \cong A$, but this statement is not true in $\mathcal{L}ens$!



- The set of elements $\mathcal{L}ens(1, A)$ provides an invariant for the category which measures the amount of **sinks** in A, the set of elements $a \in A$ such that $\sum_{x \in A} A(a, x) = 1$.
- The set of sinks is a **right adjoint** to the the discrete category functor:

$$\mathcal{L}ens(\mathrm{disc}(A), B) \cong \mathcal{S}et(A, \mathrm{sink}(B))$$

MONOIDAL CLOSEDNESS

- The hom-sets in $\mathcal{L}ens$ may be given a category structure:

$$\begin{tikzcd}
\Lambda \arrow[dd, "t"'] \arrow[dr, "\varphi"] & & \\
{} \arrow[r, phantom, "\Downarrow\tau"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \\
\Omega \arrow[ur, "\gamma"'] & &
\end{tikzcd}
=
\begin{tikzcd}
\Lambda \arrow[dd, "t"'] \arrow[dr, "\bar{\varphi}"] & \\
 & B \\
\Omega \arrow[ur, "\bar{\gamma}"'] &
\end{tikzcd}$$

$A \otimes (-)$
$\cong$

- One might suspect that $\mathcal{L}ens(A,-)$ is right adjoint to $(-)\otimes A$.
- However this is not true! Consider $A = \{\bullet \rightleftarrows \bullet\}$ and so

$$\frac{1 \otimes A = A \longrightarrow A}{A \longrightarrow \mathcal{L}ens(1,A) = \varnothing}$$

Contradiction!

- Is it possible that $(\mathcal{L}ens, \otimes, 1)$ is monoidal closed?

SUMMARY & FUTURE WORK

- In this talk, we have seen that $\mathcal{L}$ens a many "nice" aspects including:
  - pullbacks along discrete opfibs.
  - semi-cartesian monoidal structure
  - coproducts
  - distributivity and extensivity
  - equalisers
  - proper factorisation system
  - sufficient conditions for epi/mono

- What are categories internal to $\mathcal{L}$ens with source map a discrete opfibration?
- When $\mathcal{V} = \mathcal{L}$ens, is the bicategory $\mathcal{V}$-Mat interesting?
- What about $\mathcal{L}$ens-enriched cats?
- Does $\mathcal{L}$ens admit other nice factorisation systems or monoidal structures?
- Is $\mathcal{L}$ens a "nice" 2-category?