Generalising Fibrations via Multi-Valued Functions

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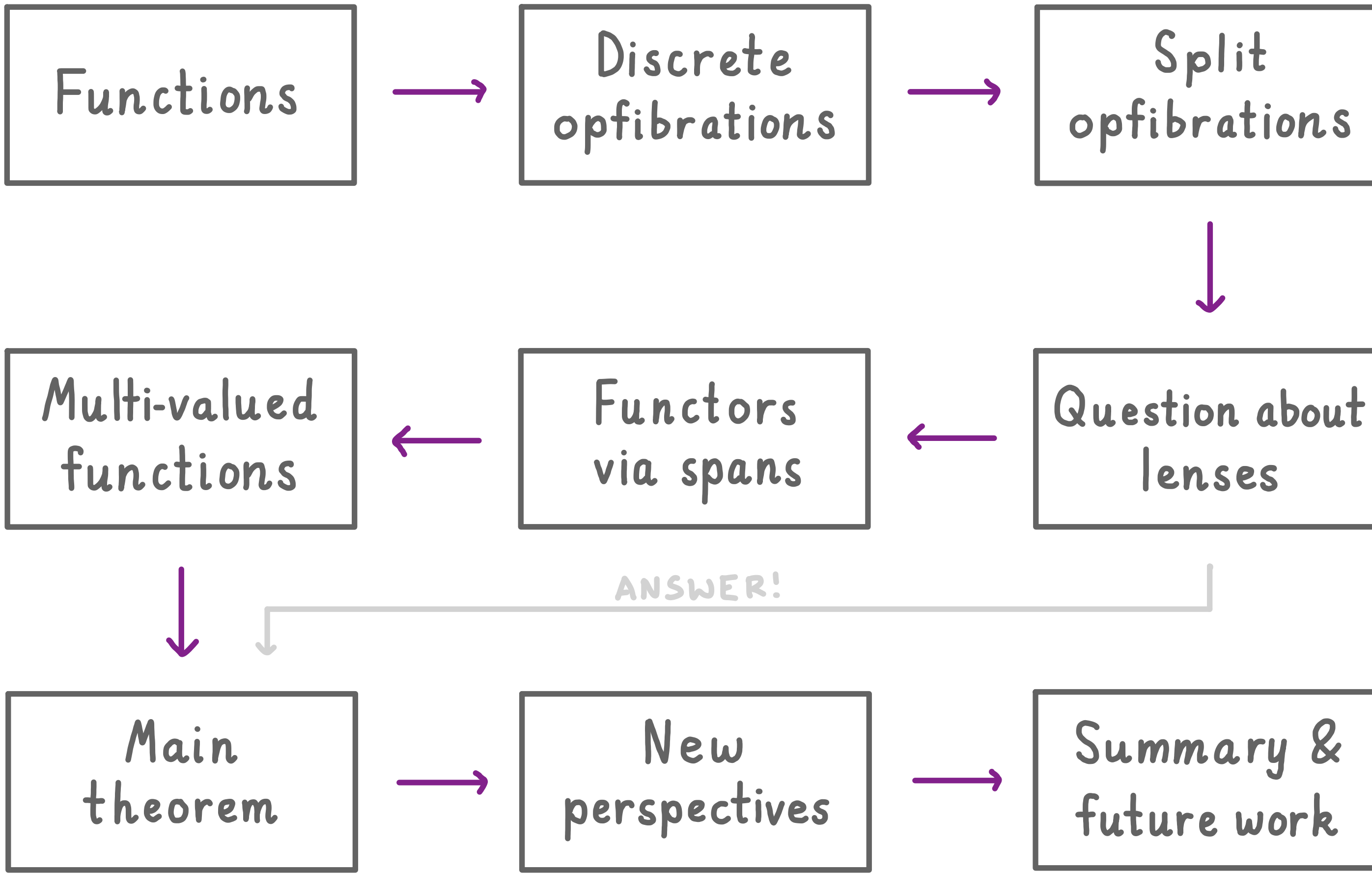
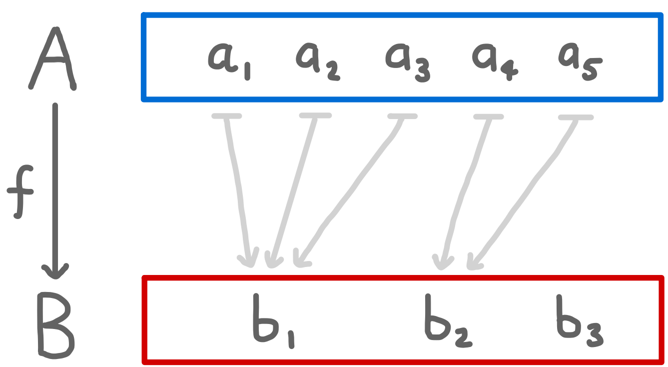
OUTLINE OF THE TALK

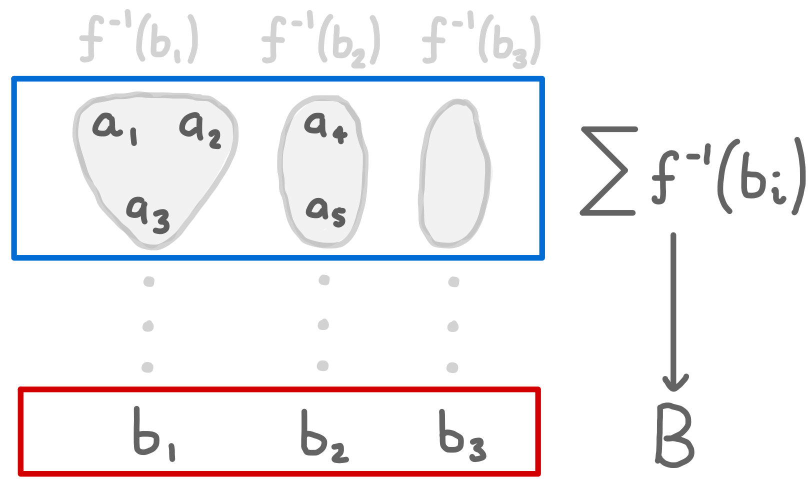


TWO PERSPECTIVES ON FUNCTIONS

For each element in the **domain**, there is an assigned element in the **codomain**.



For each element in the **codomain**, there exists a **set** called the **fibre**.



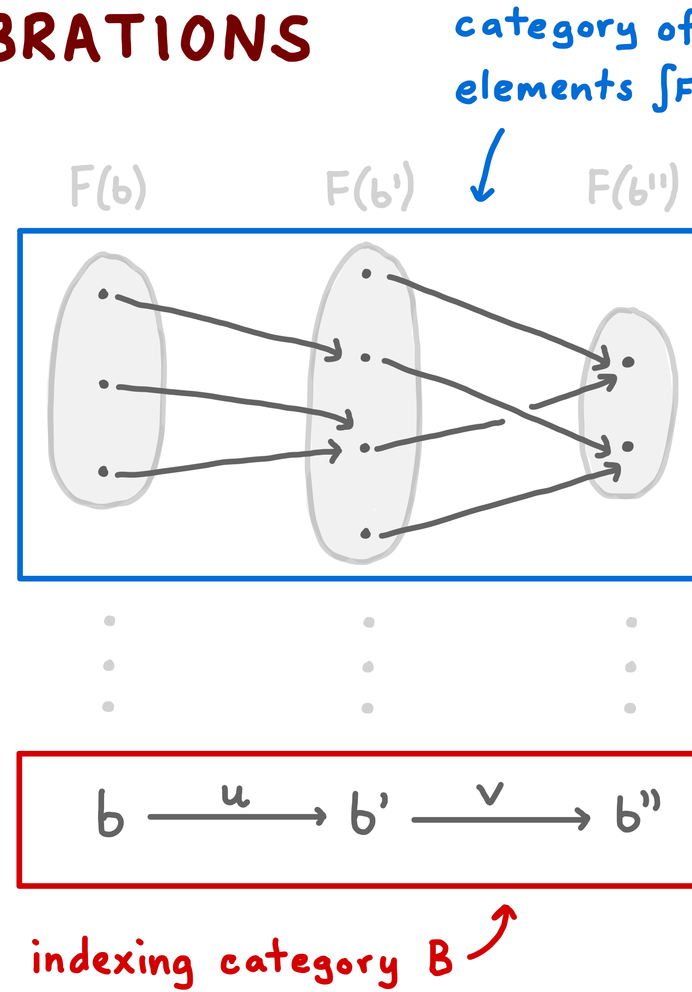
DISCRETE OPFIBRATIONS

Function $\simeq$ collection of sets indexed by a set.

Discrete opfibration $\simeq$

1) collection of sets indexed by a small category B, via a functor $F : B \longrightarrow \mathrm{Set}$.

2) functor with a certain lifting property.

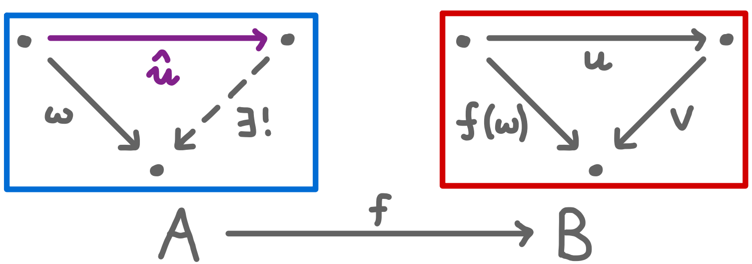


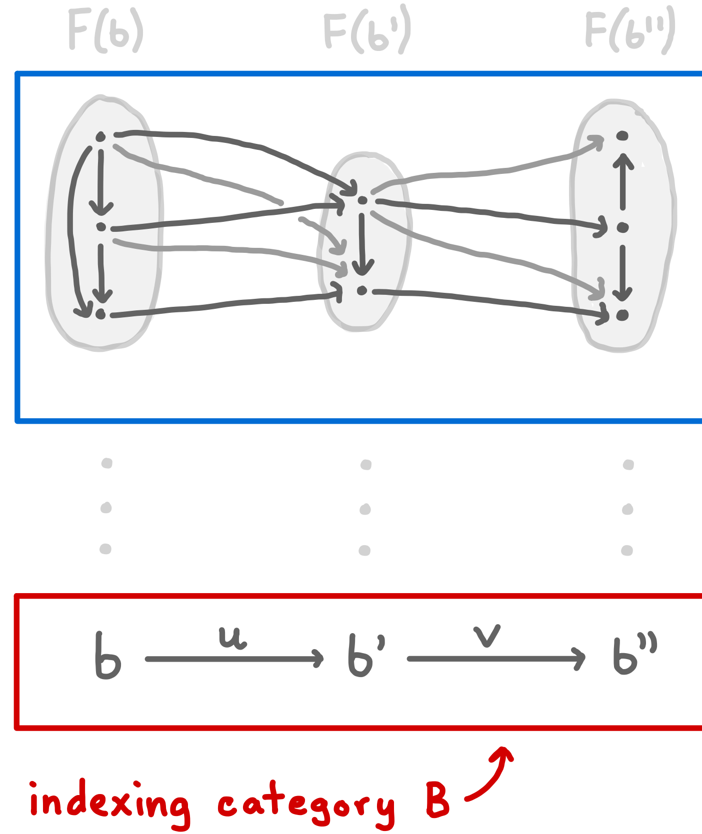
SPLIT OPFIBRATIONS

Split opfibration $\simeq$

1) collection of **small categories** indexed by a small category B, via a functor $F : B \longrightarrow \mathcal{C}at$.

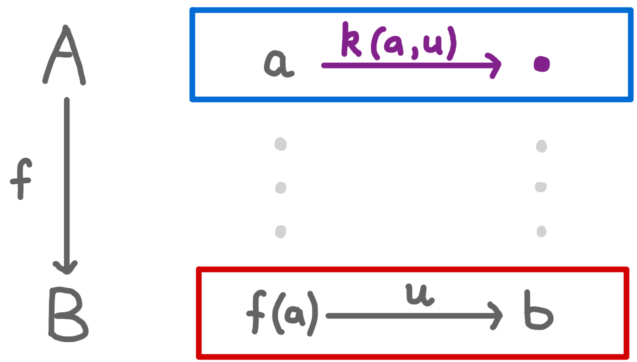
2) functor equipped with a suitable choice of **opcartesian lifts**.





A QUESTION CONCERNING LENSES

- A **lens** is a functor equipped with a suitable **choice of lifts**.



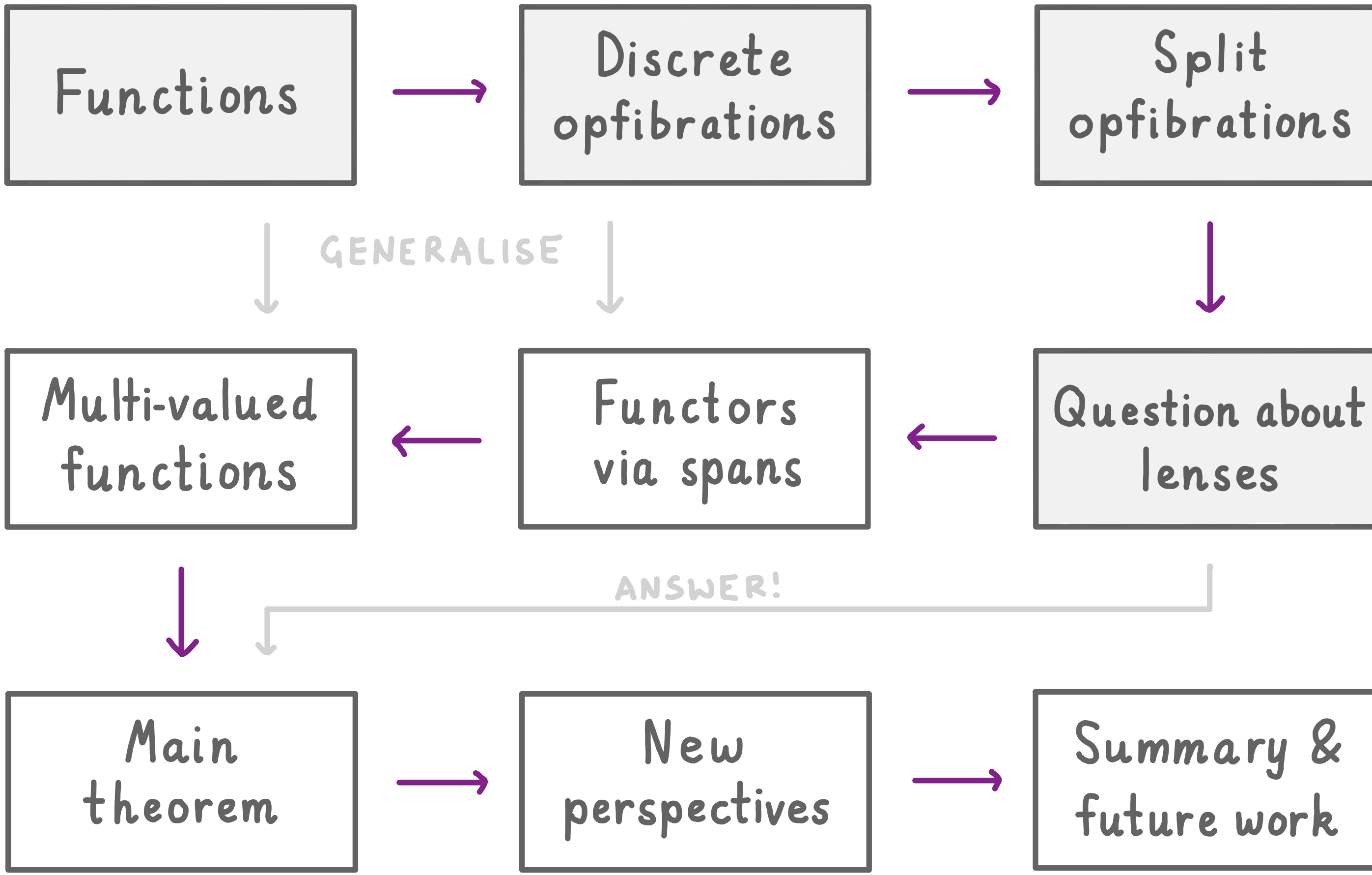
- Introduced by Diskin, Xiong, & Czarnecki in 2011 to study **bidirectional model transformations**.

- **Generalise** both discrete/split opfibrations & bijective-on-objects functors with a chosen section.

- **Natural question:** Does there exist a category $\mathcal{C}$ such that for any small category B we have an equivalence:

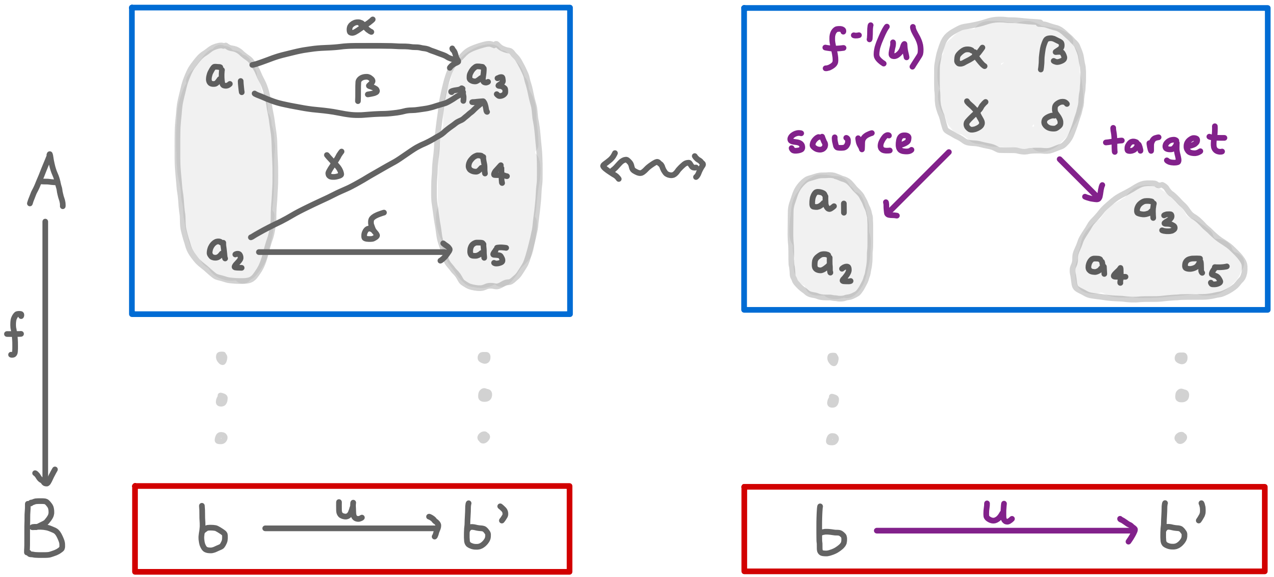
$$\frac{B \xrightarrow{F} \mathcal{C}}{\int F \longrightarrow B} \qquad \begin{array}{l}\textbf{functor}\\ \textbf{lens}\end{array}$$

OUTLINE OF THE TALK



FUNCTORS VIA SPANS OF FUNCTIONS

Idea: Functor $\simeq$ collection of sets indexed by a small category.



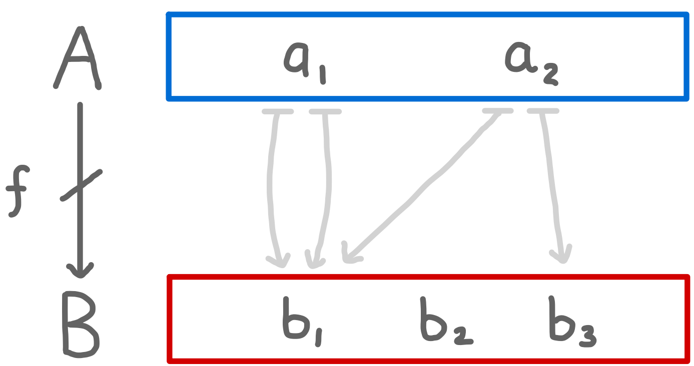
For each morphism $u \in B$, there is a span of functions between fibres.

indexing category

There is an equivalence of categories: $\mathrm{Cat}/B \simeq [B, \mathrm{Span}]_{lax}$

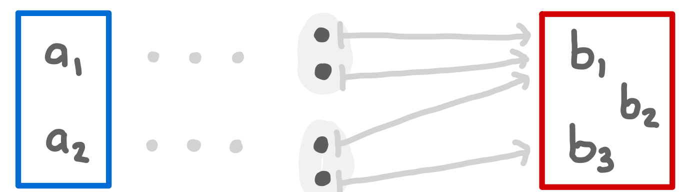
MULTI-VALUED FUNCTIONS

For each element in the domain, there is at least one assigned element in the codomain. (Not necessarily a relation!)



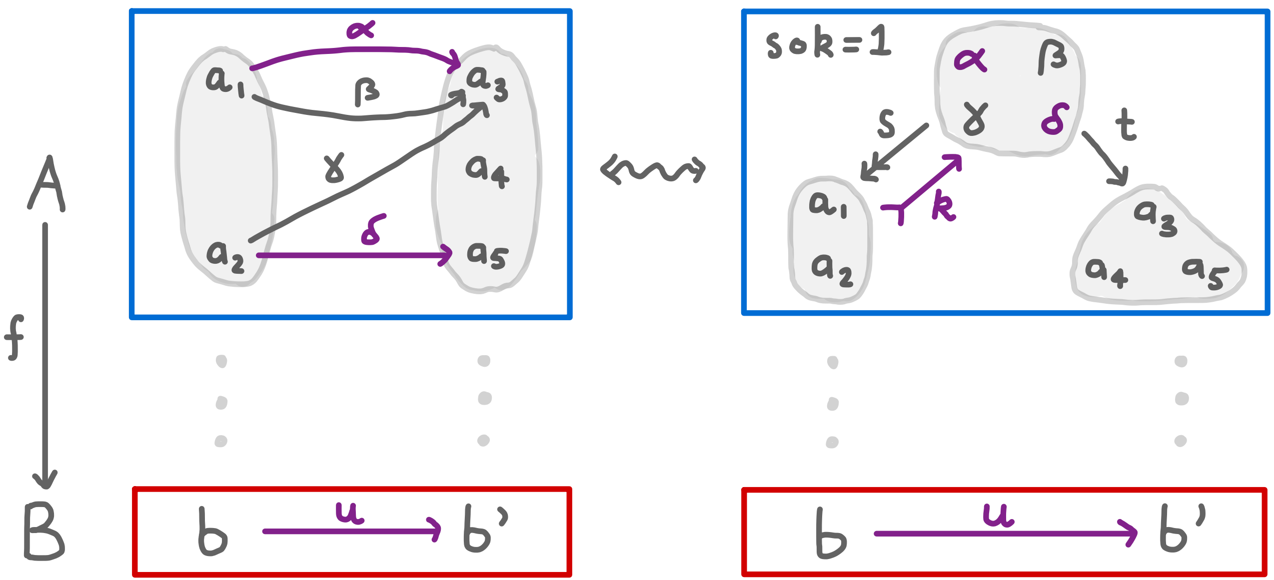
Multi-valued function $\simeq$ span of functions whose left leg is a surjection.

$$A \twoheadleftarrow \bar{A} \longrightarrow B$$



Sets and multi-valued functions form a category **Mult**.

LENSES VIA MULTI-VALUED FUNCTIONS



For each morphism $u \in B$, there is a split multi-valued function between the fibres.

Ex: Chosen lifts are $k(a_1, u) = \alpha$ and $k(a_2, u) = \delta$.

THEOREM: There is an equivalence of categories:

$$\mathcal{L}ens(B) \simeq [B, s/Mult]_{lax}$$

NEW PERSPECTIVES ON KEY EXAMPLES

- **Discrete opfibration** $\simeq$ lens such that F factors:

$$\begin{tikzcd} B \arrow[rr, "F"] \arrow[dr] & & \mathrm{sMult} \\ & \mathrm{Sq}(\mathrm{Set}) \arrow[ur] & \end{tikzcd}$$

- **Lens** $\simeq$ diagram of functors

$$\begin{tikzcd} & \Lambda \arrow[dl, "\text{b.o.o.}"'] \arrow[dr, "\text{disc. opf.}"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

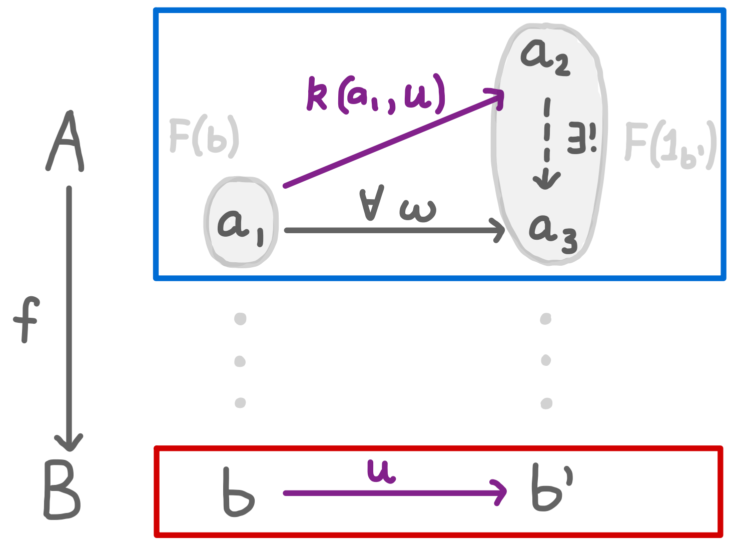
due to an **adjunction** (in Dbl):

$$\begin{tikzcd} 1 @ \mathrm{Sq}(\mathrm{Set}) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{sMult} \arrow[l, shift left] \end{tikzcd}$$

- **Split opfibration** $\simeq$ lens such that

$$F(b)\times_{F(b')}F(1_{b'}) \xrightarrow{k\times 1} F(u)\times_{F(b')}F(1_{b'}) \xrightarrow{\text{comp}} F(u)$$

is a bijection for all $u : b \to b' \in B$.



SUMMARY AND FUTURE WORK

- Lenses may be characterised via multi-valued functions:

$$\frac{B \xrightarrow{\;F\;} s\mathcal{M}\mathrm{ult}}{\int F \longrightarrow B}$$

lax double functor

lens

- Generalises similar results for:
  - functions ($B \longrightarrow \mathcal{S}\mathrm{et}$, B a set)
  - discrete opfibrations ($B \longrightarrow \mathcal{S}\mathrm{et}$)
  - split opfibrations ($B \longrightarrow \mathcal{C}\mathrm{at}$)
  - ordinary functors ($B \longrightarrow \mathcal{S}\mathrm{pan}$)

- Applications in computer science:
  - Put-based lenses (Fischer, Hu, Pacheco)
  - Supervised learning using lenses

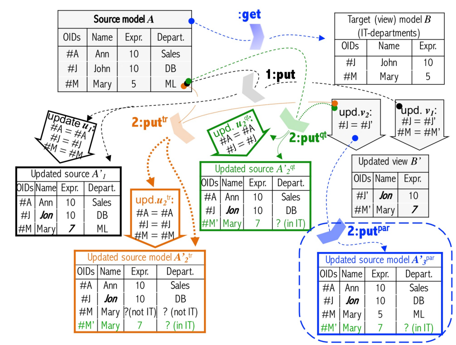


Fig. 1: Example of update propagation

An example from Diskin (2020).

- Future work in mathematics:
  - Better understand $s\mathcal{M}\mathrm{ult}$
  - Lenses as a framework for fibrations.