Cofunctors, Monoids, and Split Epimorphisms
Australian Category Seminar
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§0. Motivation

A category is like a monoid with several objects.

But what is the correct generalisation of monoid homomorphisms?

Obvious answer: A functor between categories.

Thus monoids and homomorphisms form a full subcategory of Cat:

$$\begin{tikzcd} \boxed{\mathsf{Mon}} \arrow[r, hook, "\text{f.f.}"'] & \boxed{\mathsf{Cat}} \end{tikzcd}$$

Remark: this inclusion functor has a left adjoint, which maps a small category to the free monoid on its set of morphisms modulo some relations on identities and composites; however no right adjoint exists as initial objects are not preserved.

Less obvious answer: A cofunctor between categories.

The category of small categories and cofunctors also has Mon as a full subcategory:

$$\begin{tikzcd} \boxed{\mathsf{Mon}} \arrow[r, hook, "\text{f.f.}", ""'{name=U}] & \boxed{\mathsf{Cof}} \arrow[l, bend left=0, shift left=6, "\lrcorner"', ""{name=D}] \arrow[from=U, to=D, phantom, "\perp"] \end{tikzcd}$$

This inclusion functor has a right adjoint, which maps a small category to its monoid of admissible sections.

Outline of the talk:

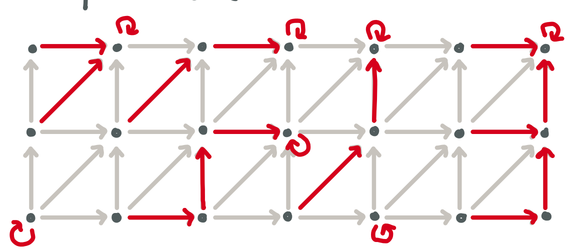
§1. The 2-category of categories and cofunctors

- What is a cofunctor?
- Examples
- Cofunctors as spans of functors
- Morphisms of cofunctors

$$
\begin{tikzcd}
 & \Lambda \arrow[dl, "\bar{\varphi}"'] \arrow[dr, "\varphi"] \arrow[dd, "t"'] & \\
B & \arrow[r, phantom, "\Downarrow\tau"] & A \\
 & \Omega \arrow[ul, "\bar{\gamma}"] \arrow[ur, "\gamma"'] &
\end{tikzcd}
$$

§2. A right adjoint: the monoid of admissible sections

- An explicit description (vector fields on categories)



- Examples
- Characterisation as a hom category
- An internal perspective

§3. Lenses and split epimorphisms of monoids

- Motivation: generalising semidirect products of groups
- Schreier split epimorphisms between monoids

$$
\begin{tikzcd}
K[f] \arrow[r, shift right, "k"'] & A \arrow[l, shift right, dashed, "q"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "\varphi"']
\end{tikzcd}
$$

- Lenses and split opfibrations between monoids

§1. The 2-category of categories and cofunctors

<u>Definition</u> (Higgins, Mackenzie ; Aguiar): A **cofunctor** $\varphi\colon B \nrightarrow A$ between small categories consists of a pair of functions,

$$\begin{array}{ccc} A_0 & \xrightarrow{\varphi_0} & B_0 \\ a & \longmapsto & \varphi_0 a \end{array}$$

assignment on objects

$$\begin{array}{ccc} A_0 \times_{B_0} B_1 & \xrightarrow{\varphi_1} & A_1 \\ (a,\ \varphi_0 a \xrightarrow{u} b) & \longmapsto & \varphi_1(a,u)\colon a \to a' \end{array}$$

assignment on morphisms

subject to the following axioms:

(1) $\varphi_0 \operatorname{cod}(\varphi_1(a,u)) = \operatorname{cod}(u)$

(2) $\varphi_1(a, 1_{\varphi_0 a}) = 1_a$

(3) $\varphi_1(a, v \circ u) = \varphi_1(a', v) \circ \varphi_1(a,u)$ where $a' = \operatorname{cod} \varphi_1(a,u)$

$$\begin{tikzcd}
A & a \arrow[r, "\varphi_1(a{,}u)"] \arrow[d, phantom, "\vdots"] & a' \arrow[d, phantom, "\vdots"] \\
B \arrow[u, "\varphi", "|" marking] & \varphi_0 a \arrow[r, "u"] & b = \varphi_0 a'
\end{tikzcd}$$

"A cofunctor is a lifting of morphisms against objects, which respects identities and composition"

Given cofunctors $\gamma\colon C \nrightarrow B$ and $\varphi\colon B \nrightarrow A$, we can take their composite in the following way:

$$
\begin{tikzcd}
A & a \arrow[rr, "\varphi_1(a,\,\gamma_1(\varphi_0 a,\,u))"] & & a' \\
B \arrow[u, "\varphi"] & \varphi_0 a \arrow[u, dotted, no head] \arrow[rr, "\gamma_1(\varphi_0 a,\,u)"] & & b \arrow[u, dotted, no head] \\
C \arrow[u, "\gamma"] & \gamma_0\varphi_0 a \arrow[u, dotted, no head] \arrow[rr, "u"] & & c \arrow[u, dotted, no head]
\end{tikzcd}
$$

This yields a category **Cof** of categories and cofunctors.

Examples:

- Every **monoid homomorphism** $B \xrightarrow{\varphi} A$ yields a cofunctor:

$$
\begin{tikzcd}
A & \bullet \arrow[r, "\varphi(m)"] & \bullet \\
B \arrow[u, "\varphi"] & \bullet \arrow[u, dotted, no head] \arrow[r, "m"] & \bullet \arrow[u, dotted, no head]
\end{tikzcd}
$$

This gives a fully faithful functor $\mathrm{Mon} \longrightarrow \mathrm{Cof}$

- More generally, every **bijective-on-objects** functor $B \longrightarrow A$ yields a cofunctor $B \nrightarrow A$.

- Every **discrete opfibration** $A \longrightarrow B$ yields a cofunctor $B \nrightarrow A$.

- Every split opfibration has an underlying cofunctor given by the splitting.

- More generally, every delta lens has an underlying cofunctor.

- Let $\mathbb{N}$ denote the monoid of natural numbers under addition. A cofunctor $\mathbb{N} \xrightarrow{\tau} A$ is the same as a choice of morphism out of every object in A:

$$
\begin{tikzcd}
A & a \arrow[r, "\tau(a,1)"] \arrow[rr, bend left, "\tau(a,2)"] & a' \arrow[r, "\tau(a',1)"] & a'' \\
 & \vdots & \vdots & \vdots \\
\mathbb{N} \arrow[uu, "\tau"] & \bullet \arrow[r, "1"] \arrow[rr, bend right, "2"'] & \bullet \arrow[r, "1"] & \bullet
\end{tikzcd}
$$

Internally this is the same as a section of the domain map:

$$
\begin{tikzcd}
A_0 \arrow[rr, "\tau"] \arrow[dr, "1"'] & & A_1 \arrow[dl, "d_0"] \\
 & A_0 &
\end{tikzcd}
$$

- For every category A, there is a unique cofunctor $\mathbb{1} \longrightarrow A$ from the category with a single identity arrow. That is, $\mathbb{1}$ is the initial object in Cof.

Proposition: Every cofunctor $B \overset{\varphi}{\nrightarrow} A$ may be represented as a span of functors,

use $\twoheadrightarrow$ for discrete opfibrations

$$\begin{tikzcd} & \Lambda \arrow[dl, "\bar{\varphi}"'] \arrow[dr, "\varphi"] & \\ B & & A \end{tikzcd}$$

where $\bar{\varphi}$ is a discrete opfibration and φ is identity-on-objects.
This representation extends to a functor $\mathsf{Cof} \longrightarrow \mathsf{Span}_{iso}(\mathsf{Cat})$, and the pair $(\mathsf{DOpf}^{op}, \mathsf{Bij})$ is an orthogonal factorisation system on Cof.

PROOF (SKETCH): Given $\varphi : B \nrightarrow A$, let Λ be the category with the same objects as A and morphisms given by formal pairs $(a, u : \varphi_0 a \longrightarrow b)$. Then we have:

$$\begin{tikzcd} B & \Lambda \arrow[l, two heads, "\bar{\varphi}"'] \arrow[r, "\varphi"] & A \end{tikzcd}$$

$$\begin{tikzcd} \varphi_0 a \arrow[d, "u"'] & a \arrow[d, "{(a,u)}"] & a \arrow[d, "{\varphi_1(a,u)}"] \\ b & a' & a' \end{tikzcd}$$

where $a' = \operatorname{cod} \varphi_1(a,u)$.
Internally this span of functors is given by the diagram:

where
$\Lambda_1 := A_0 \times_{B_0} B_1$

$$\begin{tikzcd} B_0 & A_0 \arrow[l, "\varphi_0"'] \arrow[r, "1"] & A_0 \\ B_1 \arrow[u, "d_0"] \arrow[d, "d_1"'] & \Lambda_1 \arrow[ul, phantom, very near start, "\ulcorner"] \arrow[l, "\pi_1"] \arrow[u, "\pi_0"'] \arrow[r, "\varphi_1"] \arrow[d, "d_1 \varphi_1"] & A_1 \arrow[u, "d_0"'] \arrow[d, "d_1"] \\ B_0 & A_0 \arrow[l, "\varphi_0"'] \arrow[r, "1"] & A_0 \end{tikzcd}$$

$\square$

Definition (Aguiar): A 2-cell between cofunctors,

$$\begin{tikzcd} B \arrow[r, bend left=50, "\varphi"] \arrow[r, bend right=50, "\gamma"'] \arrow[r, phantom, "\Downarrow \tau"] & A \end{tikzcd}$$

is an assignment to each object $a \in A$ of a morphism $\tau_a : a \longrightarrow a'$ in A such that $\varphi_0 a = \gamma_0 a'$ and for any pair $(a, u : \varphi_0 a \longrightarrow b)$ the following diagram in A commutes:

$$\begin{tikzcd}
& a \arrow[r, "\varphi_1(a,u)"] \arrow[d, "\tau_a"'] \arrow[dr, phantom, "\circlearrowleft"] & a'' \arrow[d, "\tau_{a''}"] & \\
A & a' \arrow[r, "\gamma_1(a',u)"'] & a''' & \\
& \vdots & \vdots & \\
& \varphi_0 a \arrow[r, "u"] \arrow[d, equal] & b \arrow[d, equal] & = \varphi_0 a'' \\
B \arrow[uuuu] & \gamma_0 a' \arrow[r, "u"'] \arrow[ur, "u"'] & \gamma_0 a''' &
\end{tikzcd}$$

where $a' = \mathrm{cod}(\tau_a)$

$a'' = \mathrm{cod}(\varphi_1(a,u))$

$a''' = \mathrm{cod}(\tau_{a''})$

$= \mathrm{cod}(\gamma_1(a',u))$

Proposition: Every 2-cell between cofunctors may be represented as a diagram in Cat:

$$\begin{tikzcd}
& \Lambda \arrow[dl, "\bar{\varphi}"'] \arrow[dr, "\varphi"] \arrow[dd, "t"'] \arrow[dd, phantom, shift left=4ex, "\Downarrow \tau"] & \\
B & & A \\
& \Omega \arrow[ul, "\bar{\gamma}"] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

$\square$

Categories, cofunctors and 2-cells form a 2-category Cof.

Vertical composition of 2-cells may be understood via pasting in Cat:

$$
\begin{tikzcd}
B \arrow[d, equal] & X \arrow[l, two heads, "\bar{\varphi}"'] \arrow[r, "\varphi"] \arrow[d, two heads, "t"'] \arrow[dr, phantom, "\Downarrow\tau"] & A \arrow[d, equal] \\
B \arrow[d, equal] & Y \arrow[l, two heads, "\bar{\gamma}"'] \arrow[r, "\gamma"] \arrow[d, two heads, "s"'] \arrow[dr, phantom, "\Downarrow\sigma"] & A \arrow[d, equal] \\
B & Z \arrow[l, two heads, "\bar{\psi}"] \arrow[r, "\psi"'] & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[d, equal] & X \arrow[l, two heads, "\bar{\varphi}"'] \arrow[r, "\varphi"] \arrow[d, two heads, "st"'] \arrow[dr, phantom, "\Downarrow\sigma t\cdot\tau"] & A \arrow[d, equal] \\
B & Z \arrow[l, two heads, "\bar{\psi}"] \arrow[r, "\psi"'] & A
\end{tikzcd}
$$

Composition of cofunctors corresponds to span composition:

$$
\begin{tikzcd}
 & & \Omega \times_B \Lambda \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & \Omega \arrow[dl, "\bar{\gamma}"'] \arrow[dr, "\gamma"] & & \Lambda \arrow[dl, two heads, "\bar{\varphi}"'] \arrow[dr, "\varphi"] & \\
C & & B & & A
\end{tikzcd}
$$

Horizontal composition of 2-cells is more involved.

$$
\begin{tikzcd}
 & W \arrow[dl, "\bar{\psi}"'] \arrow[dr, "\psi"] \arrow[dd, "s"'] & & X \arrow[dl, "\bar{\varphi}"'] \arrow[dr, "\varphi"] \arrow[dd, "t"'] & \\
C & \Downarrow\sigma & B & \Downarrow\tau & A \\
 & Z \arrow[ul, "\bar{\kappa}"] \arrow[ur, "\kappa"'] & & Y \arrow[ul, "\bar{\gamma}"] \arrow[ur, "\gamma"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & W \times_B X \arrow[dl, "\bar{\psi}\pi_W"'] \arrow[dr, "\varphi\pi_X"] \arrow[dd, "r"'] & \\
C & \Downarrow\rho & A \\
 & Z \times_B Y \arrow[ul, "\bar{\kappa}\pi_Z"] \arrow[ur, "\gamma\pi_Y"'] &
\end{tikzcd}
$$

To obtain the functor r and natural transformation ρ, first we must use the universal property of discrete opfibrations:

$$
\begin{tikzcd}
W \times_B X \arrow[rr, "\pi_X"] \arrow[d, two heads, "\pi_W"'] & & X \arrow[d, two heads, "\bar{\varphi}"] \\
W \arrow[r, two heads, "s"'] & Z \arrow[u, phantom, "\Downarrow \sigma\pi_W"] \arrow[r, "\kappa"'] & B
\end{tikzcd}
=
\begin{tikzcd}
W \times_B X \arrow[r, bend left, "\pi_X"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "\pi_W"'] & X \arrow[d, two heads, "\bar{\varphi}"] \\
W \arrow[r, "\kappa s"'] & B
\end{tikzcd}
$$

Then the functor r is constructed by the universal property of the pullback,

$$
\begin{tikzcd}
W \times_B X \arrow[r, dashed, two heads, "r"'] \arrow[rr, bend left, "th"] \arrow[d, two heads, "\pi_W"'] & Z \times_B Y \arrow[r, "\pi_Y"] \arrow[d, two heads, "\pi_Z"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads, "\bar{\gamma}"] \\
W \arrow[r, two heads, "s"'] & Z \arrow[r, "\kappa"'] & B
\end{tikzcd}
$$

while the natural transformation ρ is the horizontal composite:

$$
\begin{tikzcd}
W \times_B X \arrow[r, bend left, "\pi_X"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\theta"] & X \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\gamma t"'] \arrow[r, phantom, "\Downarrow\tau"] & A
\end{tikzcd}
$$

Note that for $\mathcal{E}$ with pullbacks, we also have a 2-category $\mathrm{Cof}(\mathcal{E})$, where a 2-cell $\varphi \overset{\tau}{\Rightarrow} \gamma : B \nrightarrow A$ may be specified by:

$$
\begin{tikzcd}
 & A_0 \arrow[r, "\varphi_0"] \arrow[dl, "1"'] \arrow[d, "\tau"] & B_0 \\
A_0 & A_1 \arrow[l, "d_0"] \arrow[r, "d_1"'] & A_0 \arrow[u, "\gamma_0"']
\end{tikzcd}
$$

+ naturality condition

§2. A right adjoint: the monoid of admissible sections

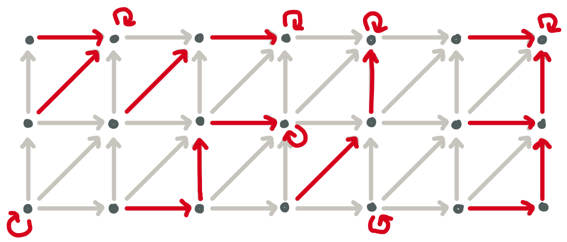
We would like to construct a right adjoint to the inclusion.

$$\begin{tikzcd} \boxed{\mathrm{Mon}} \arrow[r, hook, shift left=2, "\text{f.f.}"] \arrow[r, phantom, "\perp"] & \boxed{\mathrm{Cof}} \arrow[l, shift left=2, "\Gamma"] \end{tikzcd}$$

How can we construct a monoid from a category?

IDEA: The elements of the monoid are like vector fields on the cat.

Example (Spivak, Myers): Consider a full subcategory of $(\mathbb{N},\leq)\times(\mathbb{N},\leq)$:



At each object, choose an outgoing arrow to get a family: $(a \xrightarrow{\tau_a} t(a))_{a\in A}$

Definition: The monoid of admissible sections $\Gamma(A)$ for a category A is a monoid whose:

- elements are families of morphisms $(a \xrightarrow{\tau_a} t(a))_{a\in A}$
- unit is the family of identities $(a \xrightarrow{1_a} a)_{a\in A}$
- multiplication is given by:

$$(a \xrightarrow{\tau_a} t(a))_{a\in A} * (a \xrightarrow{\sigma_a} s(a))_{a\in A} = (a \xrightarrow{\sigma_{t(a)}\circ\tau_a} st(a))_{a\in A}$$

We will see there are several equivalent characterisations of this monoid.

Note that Γ defines the object part of the right adjoint:

$$\begin{tikzcd} \boxed{\mathrm{Set}} \arrow[r, hook, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \boxed{\mathrm{Mon}} \arrow[l, shift left=2] \arrow[r, hook, shift left=2, "\text{f.f.}"] \arrow[r, phantom, "\perp"] & \boxed{\mathrm{Cof}} \arrow[l, shift left=2, "\Gamma"] \end{tikzcd}$$

elements of the right adjoint

$$1 \longrightarrow \Gamma(A) \qquad \mathbb{N} \longrightarrow \Gamma(A) \qquad \mathbb{N} \longrightarrow A$$

admissible sections of A

Examples

- If A_0 is a discrete category, $\Gamma(A_0) \cong \mathbb{1}$

- If A is a codiscrete category, $\Gamma(A) \cong \mathrm{End}(A_0) = \mathrm{Set}(A_0, A_0)$

- If M is a monoid, then $\Gamma(M) \cong M$

- For any category A, the commutative monoid of endomorphisms of the identity transformation $\mathrm{End}(1_A) = \mathrm{Nat}(1_A, 1_A)$ is a submonoid of the centre $Z(\Gamma(A))$ of $\Gamma(A)$.

- (Garner): The group of extended inner automorphisms of a category $A \in \mathrm{Cof}$ is isomorphic to $\mathrm{Bis}(A)$, the group of bisections, which is the same as the group of invertible elements of $\Gamma(A)$.

- (Aguiar) Recall that a category in Grp is the same as a crossed module of groups. The monoid of admissible sections of such a category is equivalent to Whitehead's monoid of derivations of the crossed module.

- The 2007 paper "External derivations of internal groupoids" by Kasangian, Mantovani, Metere, and Vitale also studies this monoid.

- Every 2-cell $\tau : \varphi \Rightarrow \gamma : B \longrightarrow\!\!\!\!\!\!\!+\;\; A$ yields an element of $\Gamma(A)$.

Proposition: The monoid of admissible sections is isomorphic to the hom category:

$$\Gamma(A) := \mathrm{Cof}(\mathbb{1}, A)$$

PROOF: Consider a morphism in $\mathrm{Cof}(\mathbb{1}, A)$ given by:

$$\begin{tikzcd}
& A_0 \arrow[dl, "!"'] \arrow[dr, "i"] \arrow[dd, "t"'] & \\
\mathbb{1} & \arrow[r, phantom, "\Downarrow\tau"] & A \\
& A_0 \arrow[ul, "!"] \arrow[ur, "i"'] &
\end{tikzcd}$$

Since $\mathbb{1}$ is initial in Cof, the category $\mathrm{Cof}(\mathbb{1}, A)$ is a monoid. The components of the natural transformation τ give exactly a family of morphisms $(a \xrightarrow{\tau_a} t(a))_{a \in A}$, that is, an element of $\Gamma(A)$. Moreover, vertical composition of 2-cells corresponds exactly to composition in $\Gamma(A)$. □

Corollary: There is a functor $\Gamma : \mathrm{Cof} \longrightarrow \mathrm{Mon}$.

PROOF: Given a cofunctor $B \xrightarrow{\varphi} A$, we get a morphism of monoids by whiskering $\mathrm{Cof}(\mathbb{1}, B) \xrightarrow{\varphi} \mathrm{Cof}(\mathbb{1}, A)$. □

Note, the statement that Γ is a right adjoint translates to:

$$\mathrm{Cof}(M, A) \cong \mathrm{Mon}(M, \mathrm{Cof}(\mathbb{1}, A))$$

Is this a copower?

Proposition: There is an adjunction of categories:

$$\begin{tikzcd} \boxed{\mathbf{Mon}} \arrow[r, hook, shift left=2] \arrow[r, phantom, "\perp"] & \boxed{\mathbf{Cof}} \arrow[l, shift left=2, "\Gamma"] \end{tikzcd}$$

PROOF: The unit of the adjunction is a natural isomorphism, and the counit can be constructed as the cofunctor $\Gamma(A) \nrightarrow A$:

$$\begin{tikzcd} A & x \arrow[r, "\tau_x"] & t(x) \\ & \vdots & \vdots \\ \Gamma(A) \arrow[uu, "\varepsilon_A", "+"'] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$\left(a \xrightarrow{\tau_a} t(a)\right)_{a \in A}$$

In other words, the counit "evaluates" the family at an object $x \in A$ to get a morphism $\tau_x \in A$. One may show the required identities for an adjunction hold. $\square$

Aside: Recall admissible sections (morphisms in $\mathbf{Cof}(\mathbb{1}, A)$) are the same as cofunctors $\mathbb{N} \nrightarrow A$ (objects in $\mathbf{Cof}(\mathbb{N}, A)$). What is the relationship between these categories?

Let M be a monoid considered as a one-object category. Then we may construct a category $\hat{M}$ whose objects are morphisms in M (elements of the monoid), and whose hom sets are given by $\hat{M}(f, g) = \{m \in M \mid m \circ f = g \circ m\}$. When $M = \mathbf{Cof}(\mathbb{1}, A)$, then $\hat{M} \cong \mathbf{Cof}(\mathbb{N}, A)$. Does this construction have a universal property?

Internal perspective

Why is $\Gamma(A)$ called the monoid of admissible **sections**?

For a category $\mathcal{E}$ with pullbacks (think $\mathcal{E} = \mathrm{Set}$) we can define a functor $\Gamma : \mathrm{Cof}(\mathcal{E}) \longrightarrow \mathrm{Mon}$ which takes an internal category A to the monoid $\Gamma(A)$ whose underlying set is:

$$\Gamma(A) = \{\sigma : A_0 \longrightarrow A_1 \mid d_0 \circ \sigma = 1_{A_0}\} = \{\text{sections to the domain map } d_0\}$$

Given a cofunctor $B \overset{\varphi}{\nrightarrow} A$, we can obtain a morphism of monoids $\Gamma(B) \longrightarrow \Gamma(A)$ in the following way:

$$\Gamma(B) \overset{\varphi_0}{\rightsquigarrow} \Gamma(\Lambda) \overset{\varphi_1}{\rightsquigarrow} \Gamma(A)$$

$$\begin{tikzcd} B_0 \arrow[rr, "\sigma"] \arrow[dr, "1"'] & & B_1 \arrow[dl, "d_0"] \\ & B_0 & \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[rr, "1 \times \sigma\varphi_0"] \arrow[dr, "1"'] & & \Lambda_1 \arrow[dl, "\pi_0"] \\ & A_0 & \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[rr, "\varphi_1(1 \times \sigma\varphi_0)"] \arrow[dr, "1"'] & & A_1 \arrow[dl, "d_0"] \\ & A_0 & \end{tikzcd}$$

This is (more or less) how Aguiar originally defined Γ.

However, unlike the previous definition, Γ no longer appears as a right adjoint to the inclusion $\mathrm{Mon}(\mathcal{E}) \longrightarrow \mathrm{Cof}(\mathcal{E})$.

Can we fix this?

Assumption: Let $\mathcal{E}$ be a category with **finite limits** such that for every object $B_0 \in \mathcal{E}$, the functor

$$\mathcal{E} \xrightarrow{B_0 \times (-)} \mathcal{E}/B_0$$

has a **chosen right adjoint** $\Gamma_{B_0} : \mathcal{E}/B_0 \longrightarrow \mathcal{E}$.

Proposition: Under the assumption above, $Mon(\mathcal{E})$ is a coreflective subcategory of $Cof(\mathcal{E})$.

$$\begin{tikzcd} \boxed{Mon(\mathcal{E})} \arrow[r, hook, shift left=2, "f.f."] \arrow[r, phantom, "\perp"] & \boxed{Cof(\mathcal{E})} \arrow[l, shift left=2, "\Gamma"] \end{tikzcd}$$

PROOF: Consider an internal category $B \in Cof(\mathcal{E})$. Define:

$$\Gamma(B) := \Gamma_{B_0}\left(B_1 \xrightarrow{d_0} B_0\right) \qquad \in \mathcal{E}$$

Since $B_0 \times (-) \dashv \Gamma_{B_0}$, we have a bijection between:

$$Z \xrightarrow{\hat{\sigma}} \Gamma(B) \quad \rightsquigarrow \quad \begin{tikzcd} B_0 \times Z \arrow[rr, "\sigma"] \arrow[dr, "\pi_0"'] & & B_1 \arrow[dl, "d_0"] \\ & B_0 & \end{tikzcd}$$

In particular, when $Z = 1$, we have a correspondence between "elements" $1 \longrightarrow \Gamma(B)$ and sections of $d_0 : B_1 \longrightarrow B_0$.

We can show that $\Gamma(B)$ is an internal monoid in $\mathcal{E}$ using the diagrams for B as an internal category in $\mathcal{E}$. For example, the unit of the monoid comes from the identity map:

$$\begin{tikzcd} B_0 \arrow[rr, "i"] \arrow[dr, "1"'] & & B_1 \arrow[dl, "d_0"] \\ & B_0 & \end{tikzcd} \quad \rightsquigarrow \quad 1 \xrightarrow{e} \Gamma(B)$$

Multiplication for the monoid (plus the axioms) also follow.

To see functorality of Γ, consider a cofunctor $(\varphi_0, \varphi_1) : B \nrightarrow A$.

$$\begin{tikzcd}
A_0 \times \Gamma(B) \arrow[r, "\langle \pi_0, \varepsilon(\varphi_0 \times 1) \rangle"] \arrow[dr, "\pi_0"'] & \Lambda_1 \arrow[r, "\varphi_1"] \arrow[d, "\pi_0"] & A_1 \arrow[dl, "d_0"] & \rightsquigarrow & \Gamma(B) \arrow[r, "\Gamma_\varphi"] & \Gamma(A) \\
& A_0 & & & &
\end{tikzcd}$$

One may check that the monoid homomorphism axioms hold.

To see that the unit of the adjunction is an isomorphism, note that for a monoid M, we have a bijection:

$$\begin{tikzcd}
Z \arrow[r] & \Gamma(M) & \rightsquigarrow & Z \arrow[rr] \arrow[dr, "!"'] & & M \arrow[dl, "!"] \\
& & & & 1 &
\end{tikzcd}$$

We can easily show that $M \cong \Gamma(M)$ follows.

Finally the counit of the adjunction at an internal category B is the internal cofunctor (given as a span of internal functors) below:

$$\begin{tikzcd}
1 & B_0 \arrow[l, "!"'] \arrow[r, "1"] & B_0 \\
\Gamma(B) \arrow[u] \arrow[d] & B_0 \times \Gamma(B) \arrow[l, "\pi_1"'] \arrow[r, "\varepsilon"] \arrow[u, "\pi_0"] \arrow[d, "d_1 \varepsilon"'] & B_1 \arrow[u, "d_0"'] \arrow[d, "d_1"] \\
1 & B_0 \arrow[l, "!"'] \arrow[r, "1"] & B_0
\end{tikzcd}$$

ε is the counit of the adjunction $B_0 \times (-) \dashv \Gamma_{B_0}$

$\square$

SUMMARY (IF SHORT ON TIME)

- In §1 we saw that there is a 2-category Cof of small categories, cofunctors, and 2-cells.
- Furthermore, we saw that cofunctors and their 2-cells could be understood via diagrams in $\mathcal{C}at$.
- In §2 we proved that the category of monoids is a coreflective subcategory of Cof:

$$\begin{tikzcd} \boxed{\mathsf{Mon}} \arrow[r, hook, shift left=1.5ex, "\text{f.f.}"] \arrow[r, phantom, "\perp"] & \boxed{\mathsf{Cof}} \arrow[l, shift left=1.5ex, "\Gamma"] \end{tikzcd}$$

- We saw that the right adjoint, which takes a category to its monoid of admissible sections, has several characterisations including as the hom category $\Gamma(-) = \mathsf{Cof}(\mathbb{1}, -)$.
- Moreover, this adjunction generalises to the internal setting with suitable assumptions on $\mathcal{E}$.

DIRECTIONS FOR FUTURE WORK

- It is possible to construct a double category of functors and cofunctors from $\mathbb{S}\mathrm{pan}$ using Paré's retrocells. What can we learn from this perspective? What properties of $\mathbb{S}\mathrm{pan}$ allow us to represent cofunctors as spans?
- Replacing $\mathcal{E}$ with a suitable monoidal category $\mathcal{V}$, we can generalise the above adjunction to involve monoids and categories in $\mathcal{V}$ (this is what Aguiar did) and obtain string diagram proofs.
- Are there other examples of $\Gamma(A)$ in the literature? What interesting information about A does it contain? Applications?