Lax double functors into Span-like double categories

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Outline of the talk

0) The category of elements

1) Background on double categories

2) A generalised category of elements via lax double functors into $\mathbb{S}\mathrm{pan}$

3) Lax double functors into $\mathbb{R}\mathrm{el}$, $\mathbb{P}\mathrm{ar}$, $\mathbb{M}\mathrm{ult}$

4) Brief review of lenses

5) A certain construction on double categories

6) Lenses as lax double functors:

$$\boxed{\mathcal{L}\mathrm{ens}(B) \simeq [\mathbb{V}B, s\mathbb{M}\mathrm{ult}]_{\mathrm{lax}}}$$

7) Summary

Background: The category of elements

Given a functor $F : B \longrightarrow \mathrm{Set}$, we may construct the following comma category:

$$
\begin{tikzcd}
\int F \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\swarrow"] & 1 \arrow[d, "*"] \\
B \arrow[r, "F"'] & \mathrm{Set}
\end{tikzcd}
$$

chooses a singleton set

The **category of elements** $\int F$ has:

- objects given by pairs $(b \in B, x \in F_b)$
- morphisms $(b, x) \longrightarrow (b', x')$ given by $\beta : b \longrightarrow b' \in B$ such that $x' = F_\beta(x)$.

The resulting projection functor,

$$
\begin{aligned}
\int F &\xrightarrow{\ \pi\ } B \\
(b, x) &\longmapsto b
\end{aligned}
$$

is a **discrete opfibration**.

$$
\begin{tikzcd}
E \arrow[d, "f"'] & e \arrow[r, dashed, "\exists!"] \arrow[d, phantom, "\vdots"] & e' \arrow[d, phantom, "\vdots"] \\
B & fe \arrow[r, "\beta"'] & b
\end{tikzcd}
$$

Conversely, every discrete opfibration yields a functor into Set.

$$\mathrm{DOpf}(B) \simeq [B, \mathrm{Set}]$$

Double categories

A **double category** $\mathbb{A}$ consists of:

- a collection of objects $A, B, \ldots$
- horizontal morphisms $f : A \to B, \ldots$
- vertical morphisms $u : A \nrightarrow A', \ldots$
- cells given by diagrams:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"', "|" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "|" marking] \\ A' \arrow[r, "g"'] & B' \end{tikzcd}$$

Horizontal composition is strict, while vertical composition is associative up to comparison isocells.

Main example: $\mathbb{S}\mathrm{pan}$

- objects are sets;
- horizontal morphisms are functions;
- vertical morphisms are spans;
- cells are span morphisms:

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ X \arrow[u, "u"] \arrow[d, "u'"'] \arrow[r, "\alpha"] & Y \arrow[u, "v"'] \arrow[d, "v'"] \\ A' \arrow[r, "g"'] & B' \end{tikzcd}$$

More examples of double categories

- The terminal double category $\mathbb{1}$.

- Every category B may be made into a double category $\mathbb{V}B$ which is horizontally discrete, and a double category $\mathbb{H}B$ which is vertically discrete.

cell in $\mathbb{V}B$

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

cell in $\mathbb{H}B$

- For each category B, there is also the double category of squares $\mathbb{Q}B$, whose cells are commutative squares in B.

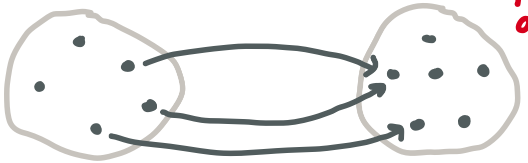
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

cell in $\mathbb{Q}B$

- $\mathbb{R}\mathrm{el}$: double category of sets, functions, and relations (jointly monic spans) with usual composition.

- $\mathbb{P}\mathrm{ar}$: double category of sets, functions, and partial functions (spans with monic left leg).

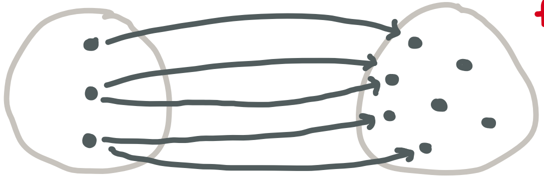
$$A \leftarrowtail X \longrightarrow B$$



partial functions are relations

- $\mathbb{M}\mathrm{ult}$: double category of sets, functions, and multi-valued functions (spans with epic left leg).

$$A \twoheadleftarrow X \longrightarrow B$$



multi-valued functions aren't always relations

Lax double functors

A **lax double functor** $F : \mathbb{A} \longrightarrow \mathbb{B}$ is given by an assignment,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ A' \arrow[r, "g"'] & B' \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "Fu"'] \arrow[dr, phantom, "F\alpha"] & FB \arrow[d, "Fv"] \\ FA' \arrow[r, "Fg"'] & FB' \end{tikzcd}$$

which preserves horizontal direction strictly, vertical direction up to comparison cells:

$$\begin{tikzcd} FA \arrow[r, equal] \arrow[d, "Fu"'] & FA \arrow[dd, "F(u \otimes v)"] \\ FB \arrow[r, phantom, "\phi(u,v)"] \arrow[d, "Fv"'] & {} \\ FC \arrow[r, equal] & FC \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, equal] \arrow[d, "1_{FA}"'] \arrow[dr, phantom, "\phi(A)"] & FA \arrow[d, "F(1_A)"] \\ FA \arrow[r, equal] & FA \end{tikzcd}$$

+ naturally & coherence conditions

- Also have colax, normal, strong, and strict double functors.
- **Example:** A lax functor $\mathbb{1} \longrightarrow \mathbb{S}\mathrm{pan}$ is the same as a small category.

$$\begin{tikzcd} A_0 \arrow[r, equal] & A_0 \\ A_1 \arrow[u, "s"] & \\ A_2 \arrow[u, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[r, "c"] & A_1 \arrow[uu, "s"'] \arrow[dd, "t"] \\ A_1 \arrow[d, "t"'] & \\ A_0 \arrow[r, equal] & A_0 \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[r, equal] & A_0 \\ A_0 \arrow[u, "1"] \arrow[d, "1"'] \arrow[r, "i"] & A_1 \arrow[u, "s"'] \arrow[d, "t"] \\ A_0 \arrow[r, equal] & A_0 \end{tikzcd}$$

composition

identities

domain

codomain

- A lax functor $\mathbb{1} \longrightarrow \mathbb{R}\mathrm{el}$ is the same as a preorder.

Horizontal transformations

A **horizontal transformation** $t : F \Rightarrow G$ between lax double functors $F, G : \mathbb{A} \to \mathbb{B}$ consists of:

- for each object A in $\mathbb{A}$, a horizontal morphism $tA : FA \longrightarrow GA$ in $\mathbb{B}$;
- for each vertical morphism $u : A \nrightarrow B$ in $\mathbb{A}$, a cell in $\mathbb{B}$:

$$\begin{tikzcd} FA \arrow[r, "tA"] \arrow[d, "Fu"', "\shortmid" marking] \arrow[dr, phantom, "tu"] & GA \arrow[d, "Gu", "\shortmid" marking] \\ FB \arrow[r, "tB"'] & GB \end{tikzcd}$$

+ naturality & coherence conditions

Proposition: Given $\mathbb{A}$ and $\mathbb{B}$, there is a category $[\mathbb{A}, \mathbb{B}]_{lax}$ whose objects are lax double functors and whose morphisms are horizontal transformations.

Corollary: There is a 2-category Dbl_{lax} of double categories with homs $[\mathbb{A}, \mathbb{B}]_{lax}$.

(we could also consider stricter versions).

Example: $[\mathbb{1}, \mathbb{S}\mathrm{pan}]_{lax} \simeq \mathcal{C}at$

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] & B_0 \\ A_1 \arrow[u, "s"] \arrow[d, "t"'] \arrow[r, "f_1"] & B_1 \arrow[u, "s"'] \arrow[d, "t"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

assignment on objects

assignment on morphisms

Category of elements for lax double functors ⑦

Consider a category B and a lax double functor $F : \mathbb{V}B \longrightarrow \mathbb{S}pan$.

- For each $b \in B$, we have a set F_b.
- For each morphism $\alpha : b \to b'$, we have a span $F_b \longleftarrow F_\alpha \longrightarrow F_{b'}$.
- Together with morphisms of spans:

$\forall b \in B$

$$\begin{tikzcd} F_b \arrow[r, equal] & F_b \\ F_b \arrow[u, "1"] \arrow[d, "1"'] \arrow[r, "i(b)"] & F_{1_b} \arrow[u, "s"'] \arrow[d, "t"] \\ F_b \arrow[r, equal] & F_b \end{tikzcd}$$

$\forall\, b \xrightarrow{\alpha} b' \xrightarrow{\beta} b'' \in B$

$$\begin{tikzcd} F_b \arrow[r, equal] & F_b \\ F_\alpha \times_{b'} F_\beta \arrow[u, "s\pi_0"] \arrow[d, "t\pi_1"'] \arrow[r, "c(\alpha,\beta)"] & F_{\beta\alpha} \arrow[u, "s"'] \arrow[d, "t"] \\ F_{b''} \arrow[r, equal] & F_{b''} \end{tikzcd}$$

We may construct the following comma:

$$\begin{tikzcd} \int F \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "*"] \\ \mathbb{V}B \arrow[r, "F"'] & \mathbb{S}pan \end{tikzcd}$$

This is a functor!

strict double functor choosing a singleton

The **category of elements** $\int F$ has:

- objects given by pairs $(b \in B, x \in F_b)$
- morphisms $(b,x) \longrightarrow (b',x')$ given by $\beta : b \longrightarrow b' \in B$ and $u \in F_\beta$ such that $s(u) = x$ and $t(u) = x'$. $(u : x \to x')$

Functors as lax double functors

Theorem: Given a category B,

$$[\mathbb{V}B, \mathbb{S}pan]_{lax} \simeq \mathcal{C}at/B.$$

Proof (sketch): For each $F : \mathbb{V}B \longrightarrow \mathbb{S}pan$ we obtain a functor,

$$\begin{tikzcd} \int F \arrow[r, "\pi"] & B \\ (b,x) \arrow[d, "{(\beta,u)}"'] & b \arrow[d, "\beta"] \\ (b',x') \arrow[r, phantom, "\longmapsto"] & b' \end{tikzcd}$$

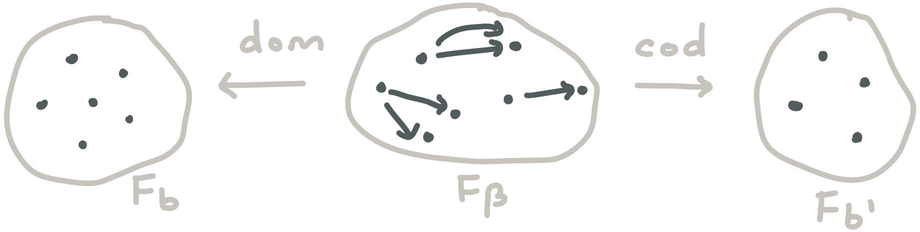
and by the universal property of the comma:

$$\begin{tikzcd} \mathbb{V}B \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathbb{S}pan \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \int F \arrow[rr, "\tau"] \arrow[dr, "\pi"'] & & \int G \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

Conversely, given a functor $f : A \longrightarrow B$, define a lax double functor $\mathbb{V}B \xrightarrow{F} \mathbb{S}pan$ via the fibre sets:

$$F_b = f^{-1}(b) = \{a \in A \mid fa = b\}$$

$$F_\beta = \{u : a \to a' \in A \mid fu = \beta : b \to b'\}$$



Functors $h : A \longrightarrow C$ such that $f = gh$ yield horizontal transformations via restrictions to the fibres $h_b : f^{-1}(b) \longrightarrow g^{-1}(b)$, etc. $\square$

See "Yoneda theory for double categories" by Paré for stronger result.

Special kinds of functors

- Normal lax functors $\mathbb{VB} \longrightarrow \mathbb{S}\mathrm{pan}$ correspond to functors with discrete fibres.
- Strong/pseudo functors $\mathbb{VB} \longrightarrow \mathbb{S}\mathrm{pan}$ correspond to discrete Conduché fibrations. That is, functors with a certain lifting property:

$$\begin{tikzcd} A \arrow[d, "f"'] & a \arrow[rr, "w"] \arrow[dr, dashed] & & a' \\ B & & x \arrow[ur, dashed] \arrow[u, phantom, "\exists!"] & \\ & fa \arrow[rr, "fw"] \arrow[dr, "u"'] & & fa' \\ & & b' \arrow[ur, "v"'] & \end{tikzcd}$$

unique factorisation lifting

- Lax functors $\mathbb{VB} \longrightarrow \mathbb{R}\mathrm{el}$ correspond to faithful functors.
- Lax functors $\mathbb{VB} \longrightarrow \mathbb{R}\mathrm{el}$ assigning each morphism $\beta : b \to b'$ to the span $F_b \xleftarrow{\pi_0} F_b \times F_{b'} \xrightarrow{\pi_1} F_{b'}$ are fully faithful.
- Lax functors $\mathbb{VB} \longrightarrow \mathbb{P}\mathrm{ar}$ correspond to faithful functors which satisfy a certain property:

$$\begin{tikzcd} & & a' \\ A \arrow[dd, "f"'] & a \arrow[ur, "u"] \arrow[dr, "v"'] & \\ & & a'' \\ B & b \arrow[r, "\beta"] & b' \end{tikzcd}$$

uniqueness of lifts

if $fu = fv = \beta$ then $u = v$.

(this implies discrete fibres)

Delta lenses (copy-pasted from previous talk)

A **(delta) lens** $(f, \varphi) : A \rightleftarrows B$ between small categories consists of a functor $f : A \longrightarrow B$ together with a function,

$$A_0 \times_{B_0} B_1 \xrightarrow{\ \varphi\ } A_1$$

$$(a, u : fa \to b) \longmapsto \varphi(a,u) : a \to p(a,u)$$

Satisfying the axioms:

(1) $f\varphi(a,u) = u$

(2) $\varphi(a, 1_{fa}) = 1_a$

(3) $\varphi(a, v \circ u) = \varphi(p(a,u), v) \circ \varphi(a,u)$

where $p(a,u) := \mathrm{cod}(\varphi(a,u))$

Proposition: Every lens can be represented as a commutative diagram of functors,

$$\begin{tikzcd}
& \Lambda \arrow[dl, "\varphi"'] \arrow[dr, "\bar{f}"] & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

where φ is bijective-on-objects and $\bar{f}$ is a discrete opfibration.

$$\begin{tikzcd}
A \arrow[d, "f"'] & a \arrow[r, "\varphi(a,u)"] \arrow[d, dotted, no head] & p(a,u) \arrow[d, dotted, no head] \\
B & fa \arrow[r, "u"'] & b
\end{tikzcd}$$

existence of chosen lifts + axioms

Lenses as lax double functors?

For each small category B, there is a category $\mathcal{L}ens(B)$ whose:

- objects are lenses with codomain B;
- morphisms are functors which make the following diagram commute:

$$\begin{tikzcd}
\Lambda \arrow[rr, "\bar{h}"] \arrow[d, "\varphi"'] \arrow[ddr, "\bar{f}"] & & \Omega \arrow[d, "\gamma"] \arrow[ddl, "\bar{g}"'] \\
A \arrow[rr, "h"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\
& B &
\end{tikzcd}$$

i.e. functors h that preserve the chosen lifts:

$h\varphi(a,u) = \gamma(ha,u)$

Central question: Does there exist a double category $\mathbb{D}$ such that:

$$\mathcal{L}ens(B) \simeq [\mathbb{V}B, \mathbb{D}]_{lax}$$

- $\mathbb{D}$ should be closely related to both $\mathbb{Q}(\mathcal{S}et)$ and $\mathbb{S}pan$, since lenses involve both discrete opfibrations and functors.
- Not every functor admits a lens structure; can we find necessary conditions?

Multi-valued functions

- Recall that a multi-valued function is given by a span of functions:

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

- A lax double functor $\mathbb{V}B \longrightarrow \mathbb{M}ult$ corresponds to functors with a certain lifting property:

$$\begin{tikzcd} A \arrow[d, "f"'] & a \arrow[r, dashed, "\exists w"] & a' \\ B & fa \arrow[r, "u"] & b \end{tikzcd}$$

existence of lifts

$\exists w$ s.t. $fw = u$.

- If there exists an opcartesian lift for each $(a, u: fa \to b)$, then f is an opfibration.

- This property is necessary for a functor to have a lens structure, but not sufficient!

- Example: Consider a functor

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \arrow[loop right, "k"] \end{tikzcd} \qquad \begin{array}{l} gf = h = h^2 \\ fg = k = k^2 \end{array}$$

$$\downarrow$$

$$\begin{tikzcd} X \arrow[loop left, "1_X"] \arrow[r, shift left, "u"] & Y \arrow[l, shift left, "v"] \arrow[loop right, "w"] \end{tikzcd} \qquad \begin{array}{l} vu = 1_X \\ uv = w \end{array}$$

By (3), $\varphi(A, v \circ u) = \varphi(B, v) \circ \varphi(A, u)$

$= g \circ f = h$

By (2), $\varphi(A, v \circ u) = \varphi(A, 1_X) = 1_A \neq h$.

Digression: A construction on double categories

Let $\mathbb{A}$ be a (unital) double category.
There is a double category $\widetilde{\mathbb{A}}$ with:

- same objects and horizontal morphisms
- vertical morphisms given by cells:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "u"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \in \mathbb{A}$$

- Vertical composition and identities:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\alpha"] & A \arrow[r, equal] \arrow[d, "u"] & A \arrow[d, "u"] \\ A \arrow[r, "f"'] \arrow[d, "1"'] & B \arrow[r, equal] \arrow[d, "1"] \arrow[dr, phantom, "\alpha'"] & B \arrow[d, "v"] \\ A \arrow[r, "f"'] & B \arrow[r, "o"'] & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "1"] & A \arrow[d, "1"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

- cells with boundary $h : A \to C$, $k : B \to D$, $\alpha : \left(A \;\overset{A}{\underset{f}{}}\; u \right)$ and $\beta : \left(C \;\overset{C}{\underset{g}{}}\; v \right)$ given by cells,

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "u"'] \arrow[dr, phantom, "\psi"] & C \arrow[d, "v"] \\ B \arrow[r, "k"'] & D \end{tikzcd} \in \mathbb{A}$$

such that

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\alpha"] & A \arrow[r, "h"] \arrow[d, "u"] \arrow[dr, phantom, "\psi"] & C \arrow[d, "v"] \\ A \arrow[r, "f"'] & B \arrow[r, "k"'] & D \end{tikzcd}$$

$$= \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "1"'] \arrow[dr, phantom, "1_h"] & C \arrow[r, equal] \arrow[d, "1"] \arrow[dr, phantom, "\beta"] & C \arrow[d, "v"] \\ A \arrow[r, "h"'] & C \arrow[r, "g"'] & D \end{tikzcd}$$

Construction (continued)

- There is a strong double functor $\widetilde{\mathbb{A}} \longrightarrow \mathbb{A}$ which is the identity on objects/horizontal morphisms:

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "u"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\longmapsto
\begin{tikzcd}
A \arrow[d, "u"] \\
B
\end{tikzcd}
$$

- There is also a strong double functor $\widetilde{\mathbb{A}} \xrightarrow{\mathcal{U}} \mathbb{Q}(\mathrm{Hor}\,\mathbb{A})$ given on vertical morphisms by:

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "u"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\longmapsto
\begin{tikzcd}
A \arrow[d, "f"] \\
B
\end{tikzcd}
$$

- If $\mathbb{A}$ has **companions**, then $\mathcal{U}$ has a left adjoint $\mathcal{F}$ which assigns each vertical arrow to its companion:

$$
\begin{tikzcd}
A \arrow[d, "f"] \\
B
\end{tikzcd}
\longmapsto
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\Theta"] & A \arrow[d, "f_*"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

- The unit for the adjunction is the identity, and the counit is given by the universal property of the companion cell:

$$
\forall \;
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "u"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad \exists! \;
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f_*"'] \arrow[dr, phantom, "\varepsilon"] & A \arrow[d, "u"] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

such that $\Theta \,|\, \varepsilon = \alpha$.

THE DOUBLE CATEGORY OF SPLIT MULTI-VALUED FUNCTIONS

We may apply our construct to $\mathbb{A} = \mathbb{S}\mathrm{pan}$.

The double category of **split multi-valued functions** $\mathrm{s}\mathbb{M}\mathrm{ult}$ has:

- objects are sets;
- horizontal morphisms are functions;
- vertical morphisms are cells in $\mathbb{S}\mathrm{pan}$:

$$\begin{tikzcd}
A \arrow[r, equal] & A \\
A \arrow[u, "1_A"] \arrow[r, "k"] \arrow[d, "1_A"'] & X \arrow[u, "s"'] \arrow[d, "t"] \\
A \arrow[r, "tk"'] & B
\end{tikzcd}$$

split epimorphism

(we really want isomorphism classes of these, but lets ignore the details)

i.e. split multi-valued functions

- Vertical composition is given by pullback:

$$\begin{tikzcd}
 & & X \times_B Y \arrow[dl, shift right, "\pi_0"'] \arrow[dr, "\pi_1"] & & \\
 & X \arrow[dl, shift right, "s"'] \arrow[dr, "t"] \arrow[ur, shift right, "\langle 1, k't \rangle"'] & & Y \arrow[dl, shift right, "s'"'] \arrow[dr, "t'"] & \\
A \arrow[ur, shift right, "k"'] & & B \arrow[ur, shift right, "k'"'] & & C
\end{tikzcd}$$

- A cell is a commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, shift right, "k"'] & A' \arrow[d, shift right, "k'"'] \\
X \arrow[u, shift right, "s"'] \arrow[r, "g"] \arrow[d, "t"'] & X' \arrow[u, shift right, "s'"'] \arrow[d, "t'"] \\
B \arrow[r, "h"'] & B'
\end{tikzcd}$$

$t'g = ht$

$s'g = fs$

$gk = k'f$

forget structure

forget property

$$\mathrm{s}\mathbb{M}\mathrm{ult} \longrightarrow \mathbb{M}\mathrm{ult} \longrightarrow \mathbb{S}\mathrm{pan}$$

An adjunction of double categories

since Span has companions

We have an adjunction of double categories (in the 2-category Dbl):

$$\begin{tikzcd} \mathbb{Q}(\mathrm{Set}) \arrow[r, hook, bend left, "\mathcal{L}"] \arrow[r, phantom, "\perp"] & \mathrm{slMult} \arrow[l, bend left, "\mathcal{R}"] \end{tikzcd}$$

The right adjoint has action on cells:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, shift right, "k"'] & A' \arrow[d, shift right, "k'"'] \\ X \arrow[r, "g"] \arrow[u, shift right, "s"'] \arrow[d, "t"'] & X' \arrow[u, shift right, "s'"'] \arrow[d, "t'"] \\ B \arrow[r, "h"'] & B' \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "tk"'] & A' \arrow[d, "t'k'"] \\ B \arrow[r, "h"'] & B' \end{tikzcd}$$

The counit for the adjunction takes a split multi-valued function to the cell:

$$\begin{tikzcd} A \arrow[r, equal] & A \arrow[d, shift right, "k"'] \\ A \arrow[u, "1_A"] \arrow[r, "k"] \arrow[d, "tk"'] & X' \arrow[u, shift right, "s"'] \arrow[d, "t"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Notice the horizontal components are identities.

Thus we have a horizontal transformation between strict double functors:

$$\begin{tikzcd} & \mathbb{Q}(\mathrm{Set}) \arrow[dr, "\mathcal{L}"] \arrow[d, phantom, "\Downarrow \varepsilon"] & \\ \mathrm{slMult} \arrow[ur, "\mathcal{R}"] \arrow[rr] & {} & \mathrm{slMult} \end{tikzcd}$$

Note that $\mathrm{DOpf}(\mathbb{B}) \simeq [\mathbb{V}\mathbb{B}, \mathbb{Q}\mathrm{Set}]$

Lenses as lax double functors into s/Mult

Theorem: Given a category B,

$$\mathcal{L}ens(B) \simeq [\mathbb{V}B, s/Mult]_{lax}$$

Proof (sketch): Given a lens $(f, \varphi) : A \rightleftarrows B$, for each $u : b \to b' \in B$, we have a span from the functor f:

$$\begin{tikzcd} & F_u \arrow[dl, "s"'] \arrow[dr, "t"] & \\ F_b & & F_{b'} \end{tikzcd}$$

But from the cofunctor part of the lens, for each $a \in F_b$ and $u : b \to b'$, there exists $\varphi(a,u) : a \to a' \in F_u$, giving the following:

$$s \circ \varphi = 1 \qquad \begin{tikzcd} & F_u \arrow[dl, shift right, "s"'] \arrow[dr, "t"] & \\ F_b \arrow[ur, shift right, "\varphi"'] & & F_{b'} \end{tikzcd}$$

The axioms of a lens ensure these split multi-valued functions behave well with identities and composition, to give a lax double functor $\mathcal{F} : \mathbb{V}B \longrightarrow s/Mult$.

Conversely, given $\mathcal{F} : \mathbb{V}B \to s/Mult$, we get a lens via the comma construction and the counit for the previous adjunction:

bijective-on-objects

$$\begin{tikzcd} \int \mathcal{L}\mathcal{R}\mathcal{F} \arrow[r] \arrow[drr, "\pi'"'] & \int \mathcal{F} \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\Swarrow"] & \mathbb{1} \arrow[d, "*"] \\ & & \\ & \mathbb{V}B \arrow[r, "\mathcal{L}\mathcal{R}\mathcal{F}"] \arrow[r, bend right, "\mathcal{F}"'] \arrow[r, phantom, shift right=3, "\Downarrow \varepsilon\mathcal{F}"] & s/Mult \end{tikzcd}$$

discrete opfibration

$\square$

Summary & Further Questions

- Discrete opfibrations are special kinds of lenses, so we were motivated to generalise the category of elements:

$$\mathrm{DOpf}(B) \simeq [B, \mathrm{Set}]$$

- We examined a generalised version involving lax double functors:

$$\mathcal{C}at/B \simeq [\mathbb{V}B, \mathbb{S}pan]_{lax}$$

- We saw how special kinds of functors could be obtained by restricting this result.
- The main result was to show:

$$\mathcal{L}ens(B) \simeq [\mathbb{V}B, s\mathbb{M}ult]_{lax}$$

- What is the category theory underlying the construction of $\widetilde{\mathbb{A}}$?
- Can we see lenses as lax <u>normal</u> double functors $\mathbb{V}B \longrightarrow \mathrm{Mod}(s\mathbb{M}ult)$?
- What are the exponentiable objects in $\mathcal{L}ens(B)$? For which B ?
- Can we characterise which lax double functors $\mathbb{V}B \to s\mathbb{M}ult$ yield split opfibrations ?
- Previously we saw that $\mathcal{L}ens(B)$ is monadic over $\mathcal{C}at/B$; can we gain a clearer perspective via the adjunction:

$$[\mathbb{V}B, s\mathbb{M}ult]_{lax} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} [\mathbb{V}B, \mathbb{S}pan]_{lax}$$

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