Lenses as algebras for a monad

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Australian Category Seminar
26 August 2020

Outline of the talk

$$\boxed{\mathcal{L}ens(B)} \xrightarrow{\mathcal{U}} \boxed{\mathcal{C}at/_B}$$

GOAL:

Show that the forgetful functor $\mathcal{U}$ is monadic

a) Construct a left adjoint $\mathcal{F}$.

b) Prove the equivalence $\mathcal{L}ens(B) \simeq (\mathcal{C}at/_B)^{\mathcal{UF}}$

PLAN:

1) Background & motivation

2) Main theorem

3) Some implications

Diskin, Xiong, Czarnecki (2011)

Background: Delta Lenses

A **(delta) lens** $(f, \varphi) : A \rightleftarrows B$ between small categories consists of a functor $f : A \longrightarrow B$ together with a function,

$$A_0 \times_{B_0} B_1 \xrightarrow{\ \varphi\ } A_1$$

$$(a, u : fa \to b) \longmapsto \varphi(a,u) : a \to p(a,u)$$

Satisfying the axioms:

(1) $f\varphi(a,u) = u$

(2) $\varphi(a, 1_{fa}) = 1_a$

(3) $\varphi(a, v \circ u) = \varphi(p(a,u), v) \circ \varphi(a,u)$

where $p(a,u) := \mathrm{cod}(\varphi(a,u))$

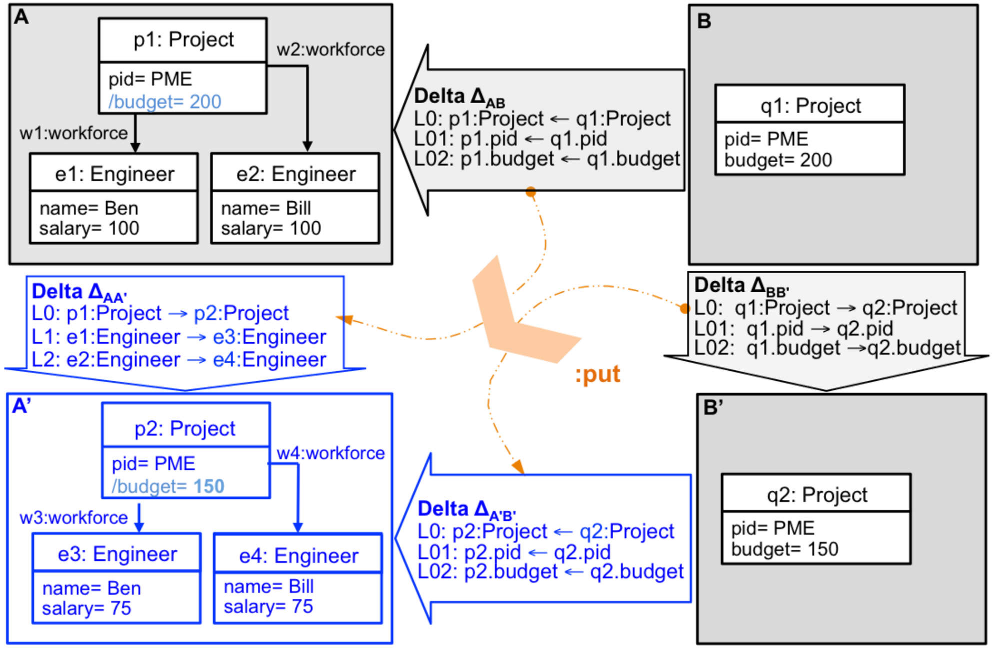
Proposition: Every lens can be represented as a commutative diagram of functors,

$$\begin{tikzcd} & \Lambda \arrow[dl, "\varphi"', "\sim"] \arrow[dr, "\bar{f}"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

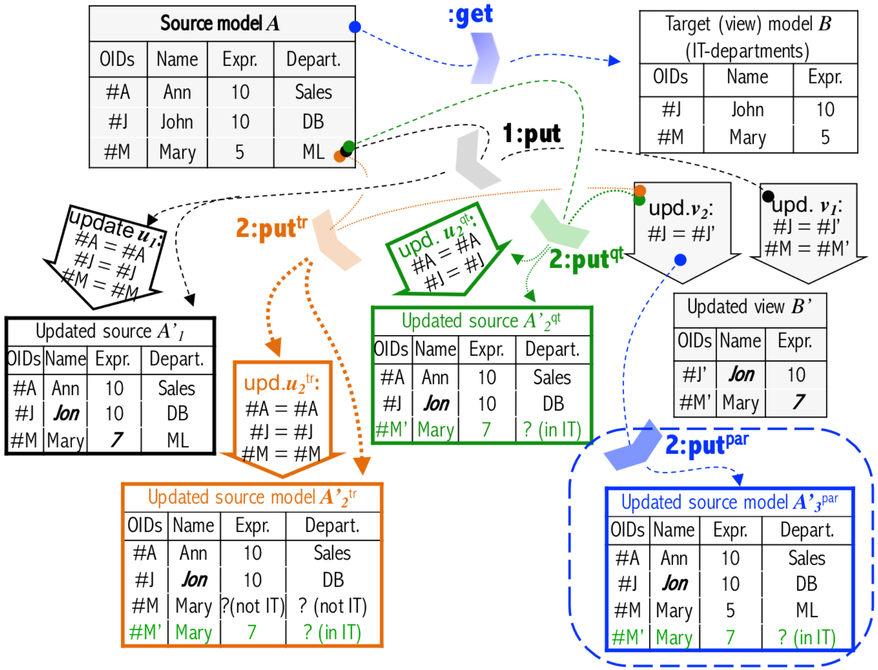
where φ is bijective-on-objects and $\bar{f}$ is a discrete opfibration.

$$\begin{tikzcd} & a \arrow[r, "(a,u)"] & p(a,u) & \\ & \arrow[dl, mapsto, "\varphi"'] & \arrow[dr, mapsto, "\bar{f}"] & \\ a \arrow[r, "\varphi(a,u)"] & p(a,u) \arrow[r, mapsto, "f"] & fa \arrow[r, "u"] & b \end{tikzcd}$$

EXAMPLES: MODEL-DRIVEN ENGINEERING



Diskin, Eramo, Pierantonio, Czarnecki (2016)



Diskin (2020)

Examples : between finite categories

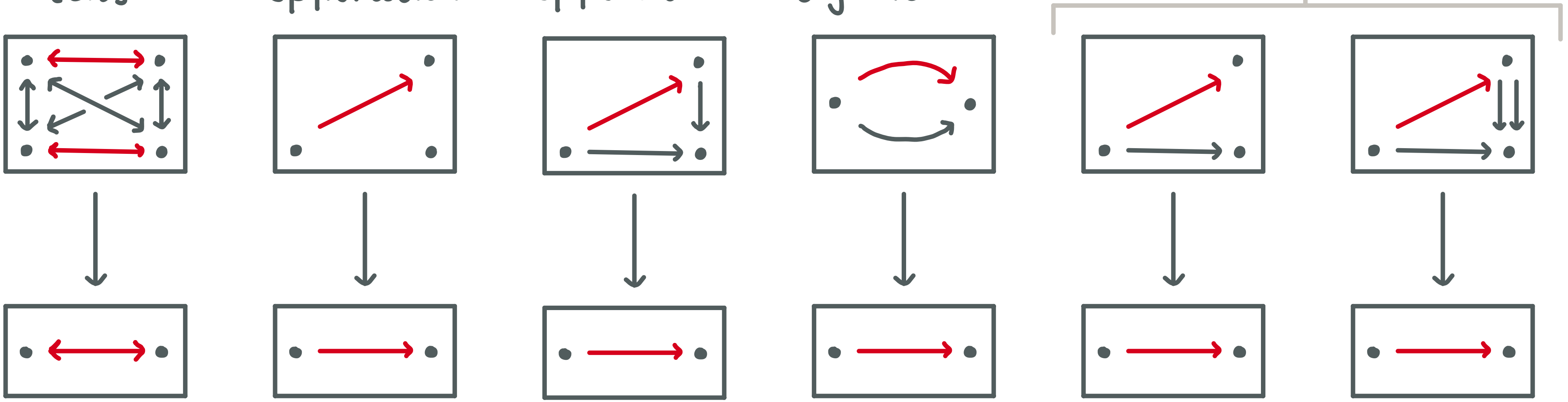
State-based lens

Discrete opfibration

Split opfibration

Bijective-on-objects lens

Lenses without opcartesian lifts



Equivalent to lens between codiscrete cats.

$$\begin{tikzcd} & A \arrow[dr, two heads] & \\ A \arrow[ur, equal] \arrow[rr, two heads] & & B \end{tikzcd}$$

Chosen lifts $\varphi(a,u)$ are opcartesian

$$\begin{tikzcd} & B \arrow[dl, "\sim"'] \arrow[dr, equal] & \\ A \arrow[rr, "\sim"'] & & B \end{tikzcd}$$

Existence of fillers fails

Uniqueness of fillers fails

State-based lenses as algebras for a monad

Oles (1982); F, G, M, P, S (2005)

$$\begin{tikzcd} \mathbf{sbLens}(B) \arrow[r, shift left, "\text{monadic}"] \arrow[r, phantom, "\top"] & \mathbf{Set}/B \arrow[l, shift left] \end{tikzcd}$$

Johnson, Rosebrugh, Wood (2010)

A **state-based lens** $(f, p) : A \rightleftarrows B$ consists of a pair of functions,

$$f : A \longrightarrow B \qquad p : A \times B \longrightarrow A$$

satisfying the axioms:

(1) $fp(a, b) = b$

(2) $p(a, fa) = a$

(3) $p(p(a, b), b') = p(a, b')$

Proposition: State-based lens are algebras for the monad:

$$\mathbf{Set}/B \longrightarrow \mathbf{Set}/B$$

$$\left(A \xrightarrow{f} B\right) \longmapsto \left(A \times B \xrightarrow{\pi} B\right)$$

- Every state-based lens is isomorphic to a free algebra.
- This generalises by replacing Set with a category with finite products.

Discrete opfibrations as algebras for a monad

full subcategory of $Cat/_B$

$$\begin{tikzcd} \boxed{DOpf(B)} \arrow[r, shift left=2, "\text{monadic}"] \arrow[r, phantom, "\top"] & \boxed{Set/_{B_0}} \arrow[l, shift left=2] \end{tikzcd}$$

$B_0 = obj(B)$

A **discrete opfibration** is a functor $f: A \longrightarrow B$ such that for all $a \in A$ and $u: fa \longrightarrow b \in B$, there exists a unique morphism $\hat{u}: a \longrightarrow a' \in A$ such that $f(\hat{u}) = u$.

$$\begin{tikzcd} A \arrow[d, "f"'] & a \arrow[r, dashed, "\hat{u}"] & a' \\ B & fa \arrow[r, "u"] & b \end{tikzcd}$$

The right adjoint above takes a discrete opfibration $f: A \longrightarrow B$ to its underlying object assignment $f_0: A_0 \longrightarrow B_0$.

The left adjoint takes a function $f_0: A_0 \longrightarrow B_0$ to the **free discrete opfibration** over B:

objects are pairs $(a, u: fa \to b)$ $\in$ $A_0 \times_{B_0} B_1$

$$\begin{tikzcd} & & & fa \arrow[dl, "u"'] \arrow[dr, "v \circ u"] & \\ f_0 \downarrow B \arrow[dd] & & b \arrow[rr, "v"'] & & b' \\ & & & & \\ B & & b \arrow[rr, "v"] & & b' \end{tikzcd}$$

- This generalises by replacing Set with a category with pullbacks.

Split opfibrations as algebras for a monad

morphisms preserve the choices of opcartesian lifts

$$\begin{tikzcd} \mathcal{S}\mathrm{Opf}(B) \arrow[r, shift left=2, "\text{monadic}"] \arrow[r, phantom, "\top"] & \mathcal{C}\mathrm{at}/B \arrow[l, shift left=2] \end{tikzcd}$$

A **split opfibration** is a lens $(f, \varphi) \colon A \rightleftarrows B$ such that for all $a \in A$ and $u \colon fa \to b \in B$, the morphism $\varphi(a, u)$ is **opcartesian**.

$$\begin{tikzcd} A \arrow[d, shift left, "(f,\varphi)"'] \\ B \arrow[u, shift left] \end{tikzcd} \qquad \begin{tikzcd} a \arrow[rr, "\varphi(a,u)"] \arrow[dr, "w"'] & & p(a,u) \arrow[dl, dashed, "\exists!"] \\ & a' & \end{tikzcd}$$

$$\begin{tikzcd} fa \arrow[rr, "u"] \arrow[dr, "fw"'] & & b \arrow[dl, "v"] \\ & fa' & \end{tikzcd}$$

The right adjoint above takes (f, φ) to $f \colon A \to B$.

The left adjoint takes a functor $f \colon A \to B$ to the **free split opfibration** over B:

$$\begin{tikzcd} & f_0{\downarrow}B \arrow[dl, "\sim"'] \arrow[dr] & \\ f{\downarrow}B \arrow[rr] & & B \end{tikzcd}$$

free discrete opfibration

Defn. in Street (1974)

comma cat. projection

Proposition: Algebras for the monad induced by the adjunction above are equivalent to split opfibrations over B.

Lenses as algebras for a... semi-monad?

$$\boxed{\mathcal{C}at/_B} \longrightarrow \boxed{\mathcal{S}et/_{B_0}} \longrightarrow \boxed{\mathrm{DOpf}(B)} \hookrightarrow \boxed{\mathcal{C}at/_B}$$

$$A \xrightarrow{f} B \;\longmapsto\; A_0 \xrightarrow{f_0} B_0 \;\longmapsto\; f_0{\downarrow}B \twoheadrightarrow B \;\longmapsto\; f_0{\downarrow}B \xrightarrow{Sf} B$$

Let's call the composite endofunctor S.

- The above endofunctor takes a functor to the free discrete opfibration on its underlying object assignment.
- There is a natural transformation $\mu : SS \Rightarrow S$ induced by the adjunction for the free discrete opfibration, whose components are given by:

$$(Sf)_0{\downarrow}B \xrightarrow{\mu_{f_0}} f_0{\downarrow}B$$

$$\begin{tikzcd} & fa \arrow[d, "u"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & b \arrow[dl, "v"'] \arrow[dr] & \\ b' \arrow[rr, "w"'] & & b'' \end{tikzcd} \;\longmapsto\; \begin{tikzcd} & fa \arrow[dl, "v \circ u"'] \arrow[dr, "w \circ v \circ u"] & \\ b' \arrow[rr, "w"'] & & b'' \end{tikzcd}$$

- The pair (S, μ) form a semi-monad on $\mathcal{C}at/_B$.

Johnson, Rosebrugh (2013)

Proposition: Lenses are equivalent to algebras for the semi-monad (S, μ) subject to an additional axiom.

$$\left(\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} ,\; \begin{tikzcd} f_0{\downarrow}B \arrow[rr, "p"] \arrow[dr, "Sf"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd} \right) \quad \text{s.t.} \quad p \circ \mu_f = p \circ S(p) \;\; \& \;\; \begin{tikzcd} A_0 \arrow[rr, "\sim"] \arrow[dr, "\hat{\mu}_f"'] & & A \\ & f_0{\downarrow}B \arrow[ur, "p"'] & \end{tikzcd}$$

extra axiom

Proof (sketch) of the equivalence

Consider a lens (f, φ) depicted by the commutative diagram:

$$\begin{tikzcd} & \Lambda \arrow[dl, "\varphi"', "\sim"] \arrow[dr, "\bar{f}"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

We can obtain an algebra for (S, μ) through taking the décalage of Λ:

$f \circ \downarrow B$
$\cong$ S

structure map

counit for $DOpf(B) \rightleftarrows Set/_{B_0}$

$$\begin{tikzcd} & D\Lambda \arrow[ddl, bend right, "\varphi \circ \varepsilon"'] \arrow[d, "\varepsilon"] \arrow[ddr, bend left, "Sf"] & \\ & \Lambda \arrow[dl, "\varphi"'] \arrow[dr, "\bar{f}"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

We may show that the pair $(f, \varphi \circ \varepsilon)$ satisfies the previous axioms.

Conversely, consider an algebra for the semi-monad given by:

$$\left(\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \;,\; \begin{tikzcd} f\circ\downarrow B \arrow[rr, "p"] \arrow[dr, "Sf"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd} \right)$$

We may define $\varphi : A_0 \times_{B_0} B_1 \longrightarrow A_1$ by :

$\varphi(a, u)$

$$p \left(\begin{tikzcd} & fa \arrow[dl, "1_{fa}"'] \arrow[dr, "u"] & \\ fa \arrow[rr, "u"'] & & b \end{tikzcd} \right) = p(\underset{\substack{\parallel \\ a}}{a, 1_{fa}}) \xrightarrow{p\langle a, u\rangle} p(a, u)$$

We may show that the lens axioms follow from the axioms for the algebra. $\boxed{/\!/}$

Goal: Lenses as algebras for a monad

$$\boxed{\mathcal{L}\mathrm{ens}(B)} \xrightarrow{\ \ \mathcal{U}\ \ } \boxed{\mathcal{C}\mathrm{at}/_B}$$

For each small category B, there is a category $\mathcal{L}\mathrm{ens}(B)$ whose:

- objects are lenses with codomain B;
- morphisms are functors which make the following diagram commute:

$$\begin{tikzcd}
\Lambda \arrow[rr, "\bar{h}"] \arrow[d, shift right, "\varphi"'] \arrow[drr, phantom, ""] \arrow[ddr, bend left=15, "\bar{f}"] & & \Omega \arrow[d, shift left, "\gamma"] \arrow[ddl, bend right=15, "\bar{g}"'] \\
A \arrow[rr, "h"] \arrow[u, shift right, "s"'] \arrow[dr, "f"'] & & C \arrow[u, shift left, "s"] \arrow[dl, "g"] \\
& B &
\end{tikzcd}$$

i.e. functors h that preserve the chosen lifts:

$$h\varphi(a,u) = \gamma(ha,u)$$

STEP 1: Construct the left adjoint $\mathcal{F} \dashv \mathcal{U}$

Proposition: The **free lens** on a functor $f : A \to B$ is constructed via the pushout:

$$\begin{tikzcd}
& A_0 \arrow[dl] \arrow[dr, "\wr"] & \\
f_0\downarrow B \arrow[dr, "\wr"] \arrow[ddr, bend right, "Sf"'] & & A \arrow[dl, "\eta_f"'] \arrow[ddl, bend left, "f"] \\
& \mathcal{F}A \arrow[d, "\mathcal{F}f"] \arrow[u, phantom, "\ulcorner"] & \\
& B &
\end{tikzcd}$$

Free lens

Garner (2019)

This describes a functor $\mathcal{F}$ which is left adjoint to $\mathcal{U}$.

A CLOSER LOOK AT THE FREE LENS

pushout injection

free discrete opfibration on $f_0 : A_0 \to B_0$

$$\begin{tikzcd} & f_0\!\downarrow\! B \arrow[dl, "\sim"'] \arrow[dr, "Sf"] & \\ FA \arrow[rr, "Ff"'] & & B \end{tikzcd}$$

The pushout FA is the category whose:

- objects are pairs $(a \in A, u : fa \to b \in B)$
- morphisms are generated by two kinds:

$$a \xrightarrow{w \in A} a' \qquad\qquad a \overset{1_a}{=\!=\!=} a$$

$$\begin{tikzcd} fa \arrow[r, "fw"] \arrow[d, equal, "1_{fa}"'] \arrow[dr, phantom, "\circlearrowleft"] & fa' \arrow[d, equal, "1_{fa'}"] \\ fa \arrow[r, "fw"'] & fa' \end{tikzcd} \qquad \text{OR} \qquad \begin{tikzcd} fa \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowleft"] & fa \arrow[d, "vu"] \\ b \arrow[r, "v \in B"'] & b' \end{tikzcd}$$

The pushout requires that:

$$1_a : (a, 1_{fa}) \longrightarrow (a, 1_{fa}) \quad \in A$$

$$\sim$$

$$1_{fa} : (a, 1_{fa}) \longrightarrow (a, 1_{fa}) \quad \in B$$

The category FA looks something like:

$$a_0 \longrightarrow a_1 \longrightarrow \cdots \longrightarrow a_m$$

$$\begin{tikzcd} fa_0 \arrow[r] \arrow[d, "1"'] & fa_1 \arrow[r] \arrow[d, "1"] \arrow[ddd, bend left] \arrow[ddddd, bend left=60] & \cdots \arrow[r] & fa_m \arrow[d, "1"] \arrow[dd, bend left] \arrow[dddd, bend left=60] \\ fa_0 \arrow[r] \arrow[d] & fa_1 \arrow[d] & & fa_m \arrow[d] \\ b_{01} \arrow[d] & b_{11} \arrow[d] & & b_{m1} \arrow[d] \\ \vdots & \vdots & & \vdots \\ b_{0i} & b_{1j} & & b_{mk} \end{tikzcd}$$

The free-forgetful adjunction for lenses

$$
\begin{tikzcd}
\boxed{\mathcal{L}ens(B)} \arrow[r, shift left=2, "\mathcal{U}"] \arrow[r, phantom, "\top"] & \boxed{\mathcal{C}at/_B} \arrow[l, shift left=2, "\mathcal{F}"]
\end{tikzcd}
$$

The component of the **counit** at a lens (f,φ) is induced by the counit from the adjunction for the free discrete opfibration:

décalage of Λ

$$
\begin{tikzcd}
f_0 \downarrow B \arrow[rr, two heads, "\varepsilon"] \arrow[d, "s"'] \arrow[ddr, bend left, "Sf"] & & \Lambda \arrow[d, "s"', "\varphi"] \arrow[ddl, bend right, "\bar{f}"'] \\
\mathcal{F}A \arrow[rr, "{[\varphi\varepsilon, 1]}"] \arrow[dr, "\mathcal{F}f"'] & & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}
$$

The counit takes $(a, 1_{fa}) \xrightarrow{\omega} (a', 1_{fa'})$ to $\omega : a \to a'$ and $(a,u) \xrightarrow{v} (a, v \circ u)$ to the chosen lift

$$p(a,u) \xrightarrow{\varphi(p(a,u),v)} p(a, v \circ u)$$

The component of the **unit** at a functor $f : A \to B$ is given by the pushout injection:

$$
\begin{tikzcd}
A \arrow[rr, "\eta_f"] \arrow[dr, "f"'] & & \mathcal{F}A \arrow[dl, "\mathcal{F}f"] \\
& B &
\end{tikzcd}
$$

The functor η_f takes $\omega : a \to a'$ to the morphism $\omega : (a, 1_{fa}) \to (a', 1_{fa'})$.

One can show that the triangle identities for an adjunction hold.

Algebras for the induced monad $\mathcal{UF}$

Let $\mathcal{UF}$ be the induced monad on $\mathcal{C}at/B$ which has assignment on objects given by:

$$A \xrightarrow{f} B \longmapsto \mathcal{F}A \xrightarrow{\mathcal{F}f} B$$

The multiplication for the monad is given by:

$$\begin{tikzcd}
& A_0 \times_{B_0} B_1 \arrow[dl] \arrow[dr, "\cong"] \arrow[dd, phantom, "\wedge", near end] & \\
(Sf)_0 \downarrow B \arrow[d, "\mu_{f_0}"'] \arrow[dr, "\cong"] & & \mathcal{F}A \arrow[dl] \arrow[ddl, bend left, "1"] \\
f_0 \downarrow B \arrow[dr, "\cong"] & \mathcal{F}\mathcal{F}A \arrow[d, "\mu_f"] & \\
& \mathcal{F}A &
\end{tikzcd}$$

discrete category

multiplication for free discrete opfibration monad

pushout injection

An **algebra for the monad** $\mathcal{UF}$ is a pair:

$$\left(\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \; , \; \begin{tikzcd} \mathcal{F}A \arrow[rr, "\hat{p}"] \arrow[dr, "\mathcal{F}f"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd} \right) \text{ s.t. } \hat{p} \circ \mu_f = \hat{p} \circ \mathcal{F}(\hat{p})$$

&

$$\begin{tikzcd}
& \mathcal{F}A \arrow[dr, "\hat{p}"] \arrow[d, phantom, "(*)"] & \\
A \arrow[ur, "\eta_f"] \arrow[rr, "1"'] & {} & A
\end{tikzcd}$$

By the universal property of the pushout, this is the same as an algebra for the semi-monad (S, μ):

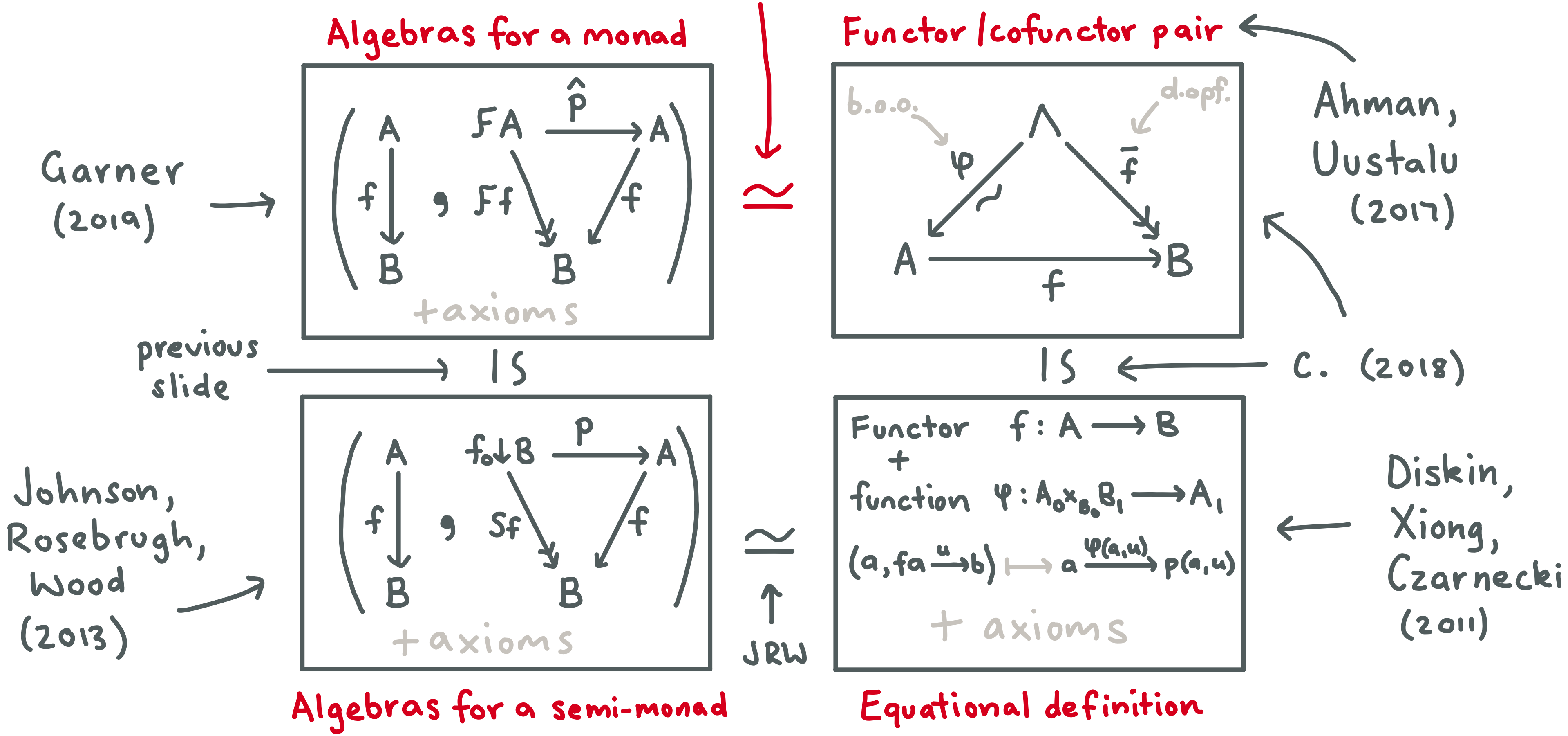
$$\left(\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \; , \; \begin{tikzcd} f_0 \downarrow B \arrow[rr, "p"] \arrow[dr, "Sf"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd} \right)$$

+ axioms

$$\begin{tikzcd}
& A_0 \arrow[dl, "\eta_{f_0}"'] \arrow[dr, "\cong"] \arrow[dd, phantom, "\wedge", near end] & \\
f_0 \downarrow B \arrow[dr, "\cong"] \arrow[ddr, bend right, "p"'] & & A \arrow[dl, "\eta_f"'] \arrow[ddl, bend left, "1"] \\
& \mathcal{F}A \arrow[d, "\hat{p}"] \arrow[r, phantom, "(*)"] & {} \\
& B &
\end{tikzcd}$$

Equivalent presentations of lenses

STEP 2: Prove the equivalence $(\mathcal{C}at_{/B})^{\mathcal{UF}} \simeq \mathcal{L}ens(B)$



A DOUBLE CATEGORY PERSPECTIVE

Given an algebra $(f, \hat{p})$ for the monad $\mathcal{U}\mathcal{F}$, there is a double category:

$$\begin{tikzcd} \mathcal{F}\mathcal{F}A \arrow[r, "\mu_f"] & \mathcal{F}A \arrow[r, shift left=2, "\hat{p}"] \arrow[r, shift right=2, "d"'] & A \arrow[l, "\eta_f" description] \end{tikzcd}$$

The domain map d is defined by:

$$\begin{tikzcd} & A_0 \arrow[dl, "\eta_{f_0}"'] \arrow[dr, "\wr"] & \\ f_0 \downarrow B \arrow[d] \arrow[dr, "\wr"'] & \wedge & A \arrow[dl, "\eta_f"] \arrow[ddl, bend left, "1"] \\ A_0 \arrow[dr, "\wr"'] & \mathcal{F}(A) \arrow[d, "d"] & \\ & A & \end{tikzcd}$$

Explicitly, the double category has:

- objects $a \in A$
- vertical morphisms $w : a \to a' \in A$
- horizontal morphisms $(a \in A, u : fa \to b \in B)$
- 2-cells are certain commutative squares in B:

$$\begin{tikzcd} a \arrow[d, "w"'] \\ a' \end{tikzcd} \qquad \begin{tikzcd} fa \arrow[r, equal] \arrow[d, "fw"'] & fa \arrow[d, "fw"] \\ fa' \arrow[r, equal] & fa' \end{tikzcd} \qquad \begin{tikzcd} a \arrow[d, "w"] \\ a' \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[d, "1_a"'] \\ a \end{tikzcd} \qquad \begin{tikzcd} fa \arrow[r, "u"] \arrow[d, equal] & b \arrow[d, "v"] \\ fa \arrow[r, "v \circ u"'] & b' \end{tikzcd} \qquad \begin{tikzcd} p(a,u) \arrow[d, "p\langle a, v\rangle"] \\ p(a, v \circ u) \end{tikzcd}$$

Comparing split opfibrations and lenses via double cats.

The **double category of squares** $Sq(B)$ for a category B has 2-cells given by all commutative squares in B:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \in B$$

However, we could restrict the 2-cells. The "**double category of triangles**" $Tri(B)$ for a category B has the same objects and vertical/horizontal morphisms as $Sq(B)$, but 2-cells are generated by commutative squares:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad \& \quad \begin{tikzcd} \bullet \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \bullet \arrow[d] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd} \in B$$

$$\begin{tikzcd} \boxed{Cat/B} \arrow[r, "\Delta_{dom}"] & \boxed{Cat/B^2} \arrow[r, "\Sigma_{cod}"] & \boxed{Cat/B} \end{tikzcd}$$

$A \xrightarrow{f} B$ $\uparrow$ pullback $\quad f{\downarrow}B \longrightarrow B^2$ $\uparrow$ compose $\quad f{\downarrow}B \longrightarrow B$

The double category of squares induces a monad above whose algebras are split opfibrations.

$$\begin{tikzcd} \boxed{Cat/B} \arrow[r, "dom"', "??"] & \boxed{Cat/B_0{\downarrow}B +_{B_0} B} \arrow[r, "\Sigma_{cod}"] & \boxed{Cat/B} \end{tikzcd}$$

$A \xrightarrow{f} B \qquad f_0{\downarrow}B +_{A_0} A \xrightarrow{[\pi, f]} B_0{\downarrow}B +_{B_0} B \qquad f_0{\downarrow}B +_{A_0} A \longrightarrow B$

The double category of triangles also induces a monad whose algebras are lenses, but is more complicated...

$$\begin{tikzcd} B_0 \arrow[r] \arrow[d] & B \arrow[d] \\ B_0{\downarrow}B \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & B_0{\downarrow}B +_{B_0} B \end{tikzcd}$$

Problem: understanding pullbacks of lenses

There is a category $\mathcal{L}ens$ whose:

- objects are small categories,
- morphisms are lenses,
- composition is given by:

$$\begin{tikzcd}
 & & \bullet \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\checkmark"] & & \\
 & \Lambda \arrow[dl, "\varphi"'] \arrow[dr, "\bar{f}"] & & \Omega \arrow[dl, "\gamma"'] \arrow[dr, "\bar{g}"] & \\
A \arrow[rr, "f"'] & & B \arrow[rr, "g"'] & & C
\end{tikzcd}$$

Motivated by the study of symmetric lenses, we would like to construct a bicategory Span ($\mathcal{L}ens$)... however $\mathcal{L}ens$ doesn't have all pullbacks!

Equivalently, for a small category B, the slice category $\mathcal{L}ens/_B$ may not have products.

Given a cospan of lenses, we can construct a canonical cone...

$$\begin{tikzcd}
\Lambda \times_B C \arrow[r, "\varphi \times 1"] \arrow[dd] & A \times_B C \arrow[ddl, "\pi_0"'] \arrow[ddr, "\pi_1"] \arrow[d, phantom, "\checkmark"] & A \times_B \Omega \arrow[l, "1 \times \gamma"'] \arrow[dd] \\
 & \, & \\
A \arrow[dr, "f"] & & C \arrow[dl, "g"'] \\
\Lambda \arrow[u, "\varphi"] \arrow[r, "\bar{f}"'] & B & \Omega \arrow[u, "\gamma"'] \arrow[l, "\bar{g}"]
\end{tikzcd}$$

This defines a semi-cartesian monoidal product on $\mathcal{L}ens/_B$.

But the universal property fails.

Not enough morphisms

Can this be fixed?

Products in the category of algebras

$$
\begin{tikzcd}
 & & \Lambda \arrow[dll, "\varphi"'] \arrow[drr, "\bar{f}"] & & \\
A \arrow[rrrr, "f"] \arrow[drr, "gf"] & & & & C \arrow[dll, "g"'] \\
\Lambda \times_C \Omega \arrow[u, "\varphi\pi_0"] \arrow[rr, "\bar{g}\pi_1"'] & & B & & \Omega \arrow[ll, "\bar{g}"] \arrow[u, "\gamma"']
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\Lambda \times_C \Omega \arrow[rr, "\pi_1"] \arrow[d, "\varphi\pi_0"'] \arrow[ddr, "\bar{g}\pi_1"] & & \Omega \arrow[d, "\gamma"] \arrow[ddl, "\bar{g}"'] \\
A \arrow[rr, "f"] \arrow[dr, "gf"'] & & C \arrow[dl, "g"] \\
 & B &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{L}ens/B \arrow[rr, "\text{bijective-on-objects}", "\text{strong monoidal}"'] \arrow[dr] & & \mathcal{L}ens(B) \arrow[dl, shift left] \\
 & \mathcal{C}at/B \arrow[ur, shift left] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}
$$

semi-cartesian monoidal category

cartesian monoidal category

strong monoidal

monadic functors create limits

cartesian monoidal category

The bicategory of spans of lenses

The bicategory of spans of lenses $\mathsf{Spn}\mathcal{L}\mathsf{ens}$ has objects given by small categories, and homs constructed by the pullback:

$$
\begin{tikzcd}
 & \mathsf{Spn}\mathcal{L}\mathsf{ens}(A,B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
\mathcal{L}\mathsf{ens}(A) \arrow[dr] \arrow[d] & & \mathcal{L}\mathsf{ens}(B) \arrow[dl] \arrow[d] \\
\mathsf{Cat}/A \arrow[r] & \mathsf{Cat} & \mathsf{Cat}/B \arrow[l]
\end{tikzcd}
$$

In other words, morphisms are spans of lenses, and 2-cells are diagrams:

$$
\begin{tikzcd}
\Omega \arrow[rr, "\gamma", "\sim"'] \arrow[dr, "\bar{g}"'] \arrow[dd] & & X \arrow[dl, "g"] \arrow[dr, "f"'] \arrow[dd] & & \Lambda \arrow[ll, "\varphi"', "\sim"] \arrow[dl, "\bar{f}"] \arrow[dd] \\
 & A & & B & \\
\Omega' \arrow[rr, "\sim", "\gamma'"'] \arrow[ur, "\bar{g}'"] & & X' \arrow[ul, "g'"'] \arrow[ur, "f'"] & & \Lambda' \arrow[ll, "\sim"', "\varphi'"] \arrow[ul, "\bar{f}'"']
\end{tikzcd}
$$

Horizontal composition is well-defined, by taking products in $\mathcal{L}\mathsf{ens}(B)$ whose projections are equipped with a lens structure via the strong monoidal functor.

$$\mathcal{L}\mathsf{ens}_{/B} \longrightarrow \mathcal{L}\mathsf{ens}(B)$$

Summary and further questions

$$\begin{tikzcd} \boxed{\mathcal{L}ens(B)} \arrow[r, shift left, "\mathcal{U}"] \arrow[r, phantom, "T"] & \boxed{\mathsf{Cat}/B} \arrow[l, shift left, "F"] \end{tikzcd}$$

- We showed that the forgetful functor from the category of lenses over B and lift-preserving functors is monadic.
- This improves a result in the literature, where lenses were equivalent to algebras for a semi-monad.
- Since $\mathcal{U}$ creates limits, $\mathcal{L}ens(B)$ has products, which can be used to construct a bicategory $\mathsf{Spn}\,\mathcal{L}ens$ whose morphisms are spans of lenses.

- The category $\mathsf{SOpf}(B)$ of split opfibrations over B and cleavage-preserving functors is a full subcategory of $\mathcal{L}ens(B)$.
- The category $\mathsf{DOpf}(B)$ of discrete opfibrations over B is a coreflective subcategory of $\mathcal{L}ens(B)$.
- These results generalise to $\mathsf{Cat}(\mathcal{E})$ for suitable $\mathcal{E}$; do they also work for more general (2-) categories?
- Does the Kleisli category for $\mathcal{U}F$ produce something interesting?
- Is the functor $\mathcal{U}$ also comonadic?
- Link with fusion for multilenses?

Another construction of Lens(B)

equaliser in CAT

$$
\begin{tikzcd}
 & \mathrm{Lens}(B) \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \bullet \arrow[dl] \arrow[dr] & \\
\mathrm{DOpf}(B) \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] \arrow[ddr, bend right] & & \mathrm{Cat}/B \arrow[dl] \arrow[ddl, bend left] \\
 & \mathrm{Cat}/B \arrow[d] & \\
 & \mathrm{Set}/B_0 &
\end{tikzcd}
$$

comma category

why should this be monadic?