A Diagrammatic Approach To Symmetric Lenses

Bryce Clarke

bryce.clarke1@hdr.mq.edu.au

MACQUARIE University
SYDNEY · AUSTRALIA

APPLIED CATEGORY THEORY 2020

Overview of the talk

$$\begin{tikzcd}
\begin{array}{c}\mathcal{D}\\ \text{discrete}\\ \text{opfibrations}\end{array} \arrow[dr] & & \begin{array}{c}\mathcal{E}\\ \text{bijective-}\\ \text{on-objects}\end{array} \arrow[dl] \arrow[dr] & & \begin{array}{c}\mathcal{M}\\ \text{fully}\\ \text{faithful}\end{array} \arrow[dl] & & \\
& \mathbf{Cof} \arrow[dl] \arrow[dr] & & \mathbf{Cat} \arrow[dl] \arrow[rr, squiggly] & & \mathbf{Lens} \arrow[dl] \arrow[dr] & \\
\mathbf{Span}(\mathbf{Cat}) & & \mathbf{Meal} \arrow[ll] \arrow[rr, squiggly] & & \mathbf{SymLens} \arrow[rr, shift left=3] \arrow[rr, shift right=3] \arrow[rr, phantom, shift left=1.5, "\perp"] \arrow[rr, phantom, shift right=1.5, "\perp"] & & \mathbf{SpnLens} \arrow[ll]
\end{tikzcd}$$

GOAL 1: Develop a diagrammatic framework for lenses.

GOAL 2: Understand the relationship between symmetric & asymmetric lenses.

WHAT IS A LENS?

CATEGORY
objects = states
morphisms = updates

$$\begin{tikzcd} \text{SYSTEM A} \arrow[r, shift left] & \text{SYSTEM B} \arrow[l, shift left, "\text{LENS}"] \end{tikzcd}$$

"maintains consistency between states of systems"

ASYMMETRIC LENS

$$A \xrightarrow{f} B$$

GET (functor)

$$\begin{tikzcd} a \arrow[d, "\omega"'] \arrow[r, phantom, "\cdots"] & fa \arrow[d, "f\omega"] \\ a' \arrow[r, phantom, "\cdots"] & fa' \end{tikzcd}$$

PUT (cofunctor)

$$\begin{tikzcd} a \arrow[d, "\varphi(a,u)"'] \arrow[r, phantom, "\cdots"] & fa \arrow[d, "u = f\varphi(a,u)"] \\ a' \arrow[r, phantom, "\cdots"] & b = fa' \end{tikzcd}$$

SYMMETRIC LENS

$$A \xleftarrow{g_0} X_0 \xrightarrow{f_0} B$$

$$\begin{tikzcd} g_0x \arrow[d, "\omega"'] \arrow[r, phantom, "\cdots"] & x \arrow[r, phantom, "\cdots"] & f_0x \arrow[d, "f_1(x,\omega)"] \\ g_0x' = a \arrow[r, phantom, "\cdots"] & x' \arrow[r, phantom, "\cdots"] & f_0x' \end{tikzcd}$$

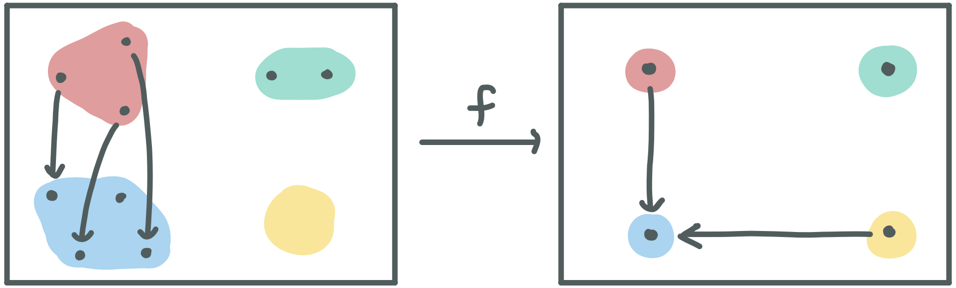
$$\begin{tikzcd} g_0x \arrow[d, "g_1(x,u)"'] \arrow[r, phantom, "\cdots"] & x \arrow[r, phantom, "\cdots"] & f_0x \arrow[d, "u"] \\ g_0x'' \arrow[r, phantom, "\cdots"] & x'' \arrow[r, phantom, "\cdots"] & b = f_0x'' \end{tikzcd}$$

Three classes of functors

$\mathcal{D}$

discrete opfibrations

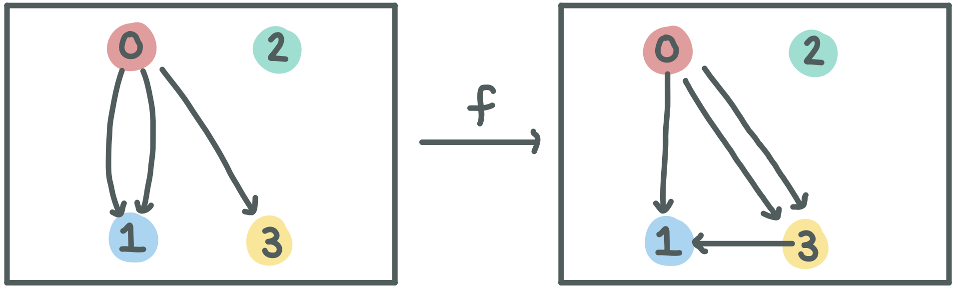
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ a & \cdots & fa \\ {\scriptstyle \exists! w}\downarrow & & \downarrow{\scriptstyle u = fw} \\ a' & \cdots & b = fa' \end{array}$$



$\mathcal{E}$

bijective-on-objects

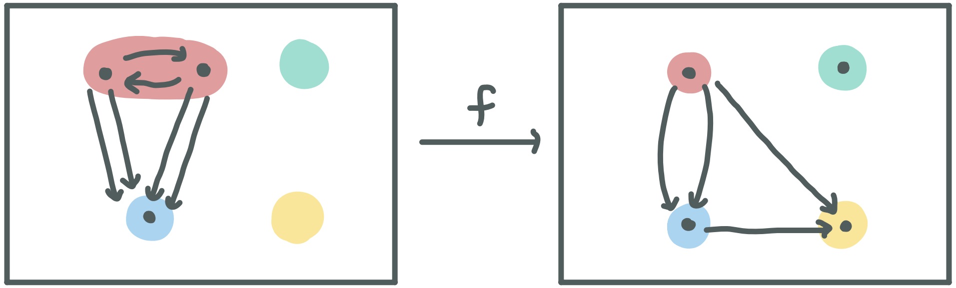
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \exists!\ a & \cdots & b = fa \end{array}$$



$\mathcal{M}$

fully faithful

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ a & \cdots & fa \\ {\scriptstyle \exists! w}\downarrow & & \downarrow{\scriptstyle u = fw} \\ a' & \cdots & fa' \end{array}$$



Cofunctors & factorisation systems

Cof
small categories & cofunctors

$$\begin{tikzcd} A & & B \arrow[ll, "\varphi"'] \\ a \arrow[d, "\varphi_1(a,u)"'] \arrow[rr, phantom, "\cdots"] & & \varphi_0 a \arrow[d, "u"] \\ a' \arrow[rr, phantom, "\cdots"] & & b = \varphi_0 a' \end{tikzcd}$$

"each update $u : \varphi_0 a \longrightarrow b \in B$ has a chosen lift"

+

respects identities and composition

Cat
small categories & functors

factorisation system $\rightsquigarrow$

$\mathcal{E}$
bijective-on-objects

+

$\mathcal{M}$
fully faithful

$$\begin{tikzcd} & I \arrow[dr, "m"] & \\ A \arrow[ur, "e"] \arrow[rr, "f"'] & & B \end{tikzcd}$$

Cof
small categories & cofunctors

factorisation system $\rightsquigarrow$

$\mathcal{D}^{op}$
discrete opfibrations

+

$\mathcal{E}$
bijective-on-objects

$$\begin{tikzcd} & \Lambda \arrow[dl, "d"'] \arrow[dr, "e"] & \\ B \arrow[rr, "\varphi"'] & & A \end{tikzcd}$$

THE BICATEGORY OF MEALY MORPHISMS

$\boxed{\begin{array}{c}\mathbf{Meal}\\ \text{small categories,}\\ \text{Mealy mor. \& 2-cells}\end{array}}$

$$\begin{tikzcd} & X \arrow[dl, "d"'] \arrow[dr, "f"] & \\ A \arrow[rr, "|"] & & B \end{tikzcd}$$

$d \in \mathcal{D}$

"partial map between categories"

$$\begin{tikzcd} A & X \arrow[l, "d"'] \arrow[r, "f"] & B \end{tikzcd}$$

$$\begin{tikzcd} d_0x \arrow[d, "\omega"'] & \cdots & x \arrow[d, "(x,\omega)"] & \cdots & f_0x \arrow[d, "f_1(x,\omega)"] \\ d_0x' = a & \cdots & x' & \cdots & f_0x' \end{tikzcd}$$

EXAMPLE: MEALY MACHINES

$f : X \times A \longrightarrow B$

OUTPUT

$t : X \times A \longrightarrow X$

TRANSITION

"Mealy morphism between free monoids"

$$\begin{tikzcd} A^* & X \arrow[l] \arrow[r] & B^* \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[d, "a_1"'] & \cdots & x_0 \arrow[d] & \cdots & \bullet \arrow[d, "f(x_0,a_1)"] \\ \bullet & \cdots & t(x_0,a_1) & \cdots & \bullet \end{tikzcd}$$

$\boxed{\begin{array}{c}\mathbf{Meal}\\ \text{small categories,}\\ \text{Mealy mor. \& 2-cells}\end{array}}$ factorisation system $\rightsquigarrow$ $\overbrace{\boxed{\begin{array}{c}\mathcal{D}^{op}\\ \text{discrete opfibrations}\end{array}} + \boxed{\begin{array}{c}\mathcal{E}\\ \text{bijective-on-objects}\end{array}}}^{\mathbf{Cof}}$ $\underbrace{\phantom{\mathcal{E}} + \boxed{\begin{array}{c}\mathcal{M}\\ \text{fully faithful}\end{array}}}_{\mathbf{Cat}}$

$\simeq \mathrm{Mnd}(\mathrm{Span})$

Asymmetric lenses

Cof + **Cat** $\rightsquigarrow$ **Lens**

"PUT" (Cof) — "GET" (Cat)

Lens: Small categories & asym. lenses

$$\begin{tikzcd} & \Lambda \arrow[dl, "\varphi"'] \arrow[dr, "\bar{f}"] \arrow[d, phantom, "\circlearrowright"] & \\ A \arrow[rr, "f"'] & {} & B \end{tikzcd}$$

$\varphi \in \mathcal{E}$

$\bar{f} \in \mathcal{D}$

"functor with a suitable choice of lifts"

EXAMPLES

- A, B codiscrete $\leftrightsquigarrow$ very well-behaved lenses

$f : A \longrightarrow B$ $\qquad$ $p : A \times B \longrightarrow A$

(PUT-GET) $\quad fp(a,b) = b$

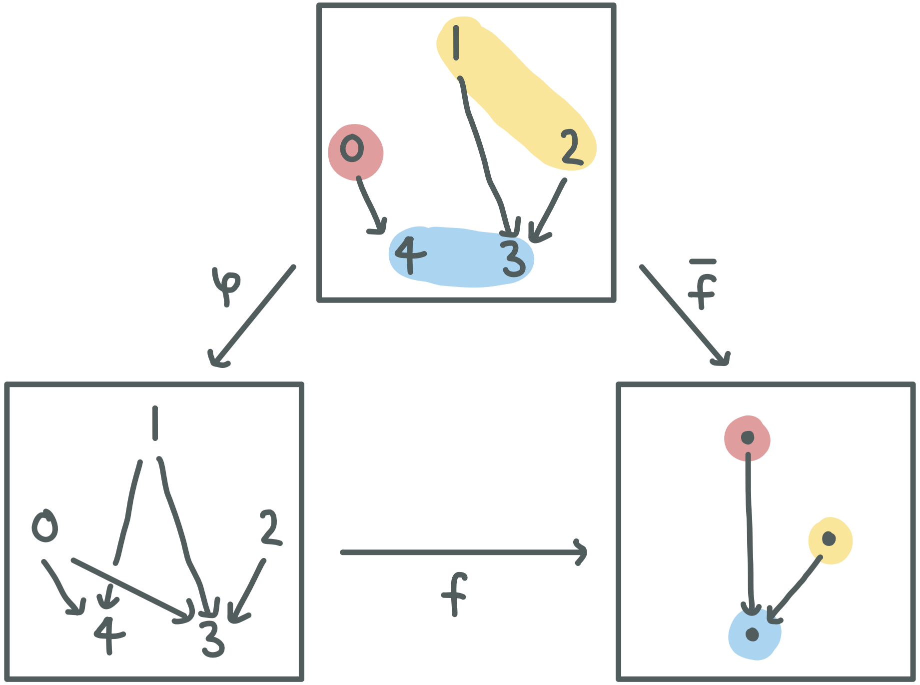
(GET-PUT) $\quad p(a, fa) = a$

(PUT-PUT) $\quad p(p(a,b), b') = p(a, b')$

LENS LAWS

- Split opfibrations $\leftrightsquigarrow$ $B \longrightarrow \mathcal{C}at$
- A, B monoids $\leftrightsquigarrow$ section/retraction

$$\begin{tikzcd} B \arrow[r, "\varphi"'] \arrow[rr, bend left, "1_B"] & A \arrow[r, "f"'] & B \end{tikzcd}$$



The bicategory of spans of asymmetric lenses

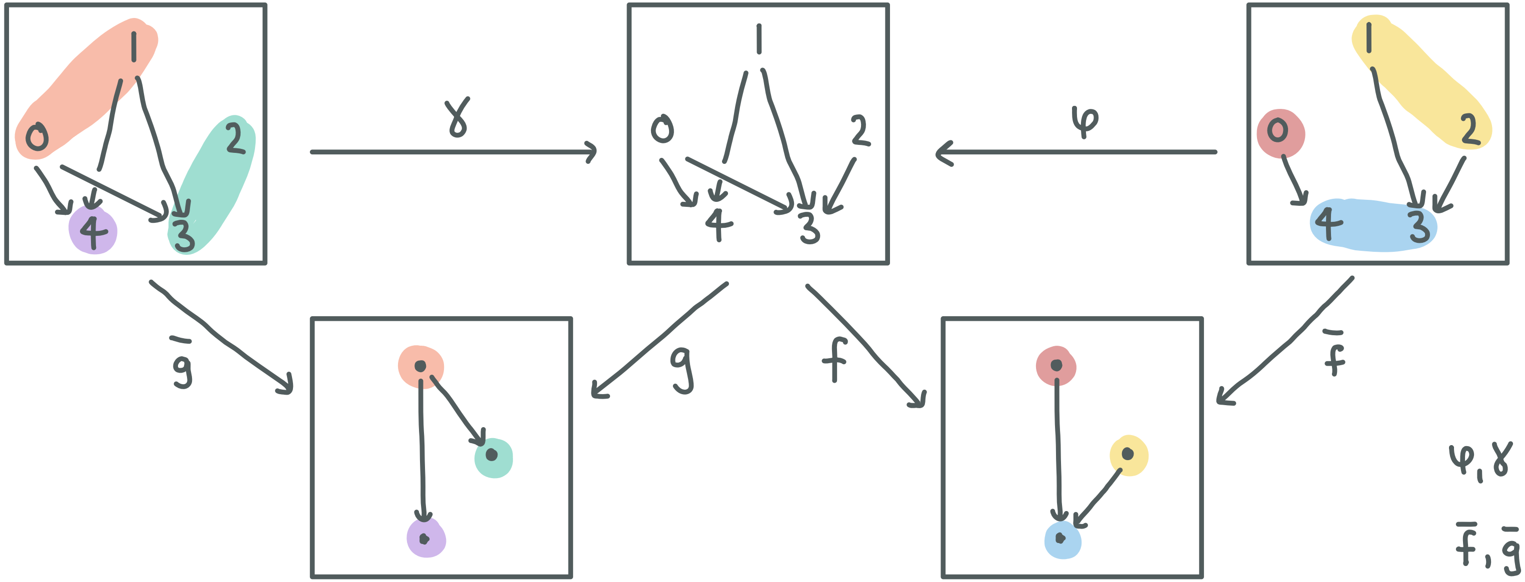
$$\begin{tikzcd} \mathcal{L}ens \arrow[d] \\ \mathcal{S}pn\mathcal{L}ens \end{tikzcd}$$

$$\begin{tikzcd} \Omega \arrow[rr, "\gamma"] \arrow[dr, "\bar{g}"'] & & X \arrow[dl, "g"] \arrow[dr, "f"'] & & \Lambda \arrow[ll, "\varphi"'] \arrow[dl, "\bar{f}"] \\ & A & & B & \end{tikzcd}$$

1-cells

$$\begin{tikzcd} \Omega \arrow[rr, "\gamma"] \arrow[dr, "\bar{g}"] \arrow[dd] & & X \arrow[dl, "g"] \arrow[dr, "f"] \arrow[dd] & & \Lambda \arrow[ll, "\varphi"'] \arrow[dl, "\bar{f}"] \arrow[dd] \\ & A & & B & \\ \Omega' \arrow[rr, "\gamma'"'] \arrow[ur, "\bar{g}'"] & & X' \arrow[ul, "g'"] \arrow[ur, "f'"] & & \Lambda' \arrow[ll, "\varphi'"] \arrow[ul, "\bar{f}'"] \end{tikzcd}$$

2-cells



$\varphi, \gamma \in \mathcal{E}$

$\bar{f}, \bar{g} \in \mathcal{D}$

The bicategory of symmetric lenses

Lens

$\downarrow$

$\boxed{\text{SymLens}}$

"suitable pair of Mealy morphisms"

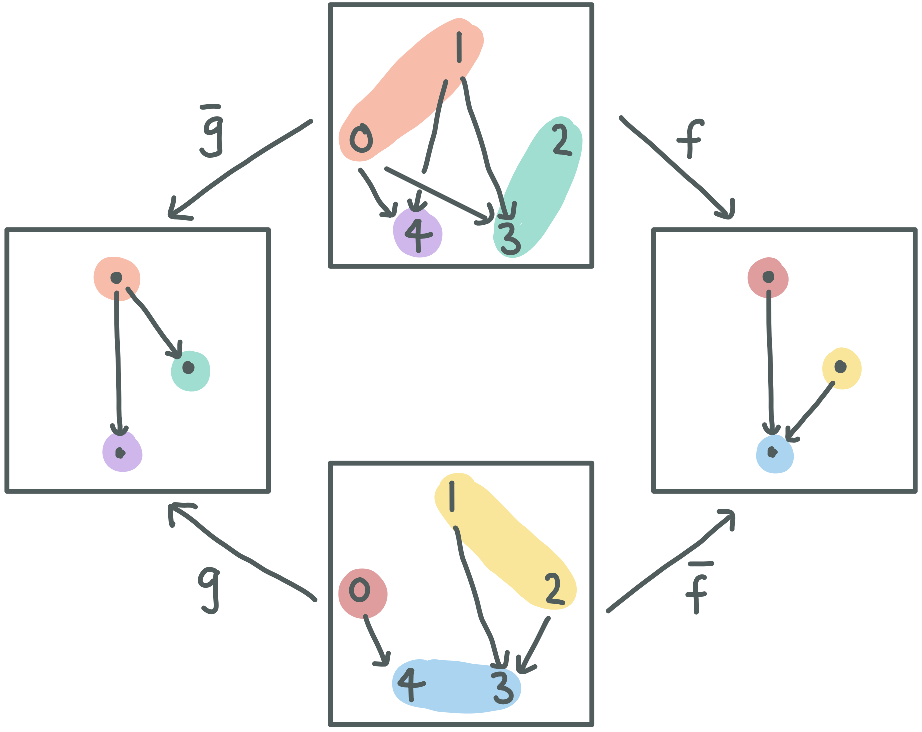
$$\begin{tikzcd}
& X^+ \arrow[dl, "\bar{g}"'] \arrow[dr, "f"] & \\
A & X_0 \arrow[l, "g_0"'] \arrow[r, "f_0"] \arrow[u] \arrow[d] & B \\
& X^- \arrow[ul, "g"] \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

1-cells

$\bar{g}, \bar{f} \in \mathcal{D}$

$$\boxed{\text{SpnLens}(A,B)} \xrightarrow[\mathcal{M}]{\text{"compose"}} \boxed{\text{SymLens}(A,B)}$$

$$\begin{tikzcd}
\Omega \arrow[r, "\gamma"] \arrow[dr, "\bar{g}"'] & X \arrow[d, "g"'] \arrow[dr, "f"] & \Lambda \arrow[l, "\varphi"'] \arrow[d, "\bar{f}"] \\
& A & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
& \Omega \arrow[dl, "\bar{g}"'] \arrow[dr, "f\gamma"] & \\
A & & B \\
& \Lambda \arrow[ul, "g\varphi"] \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$



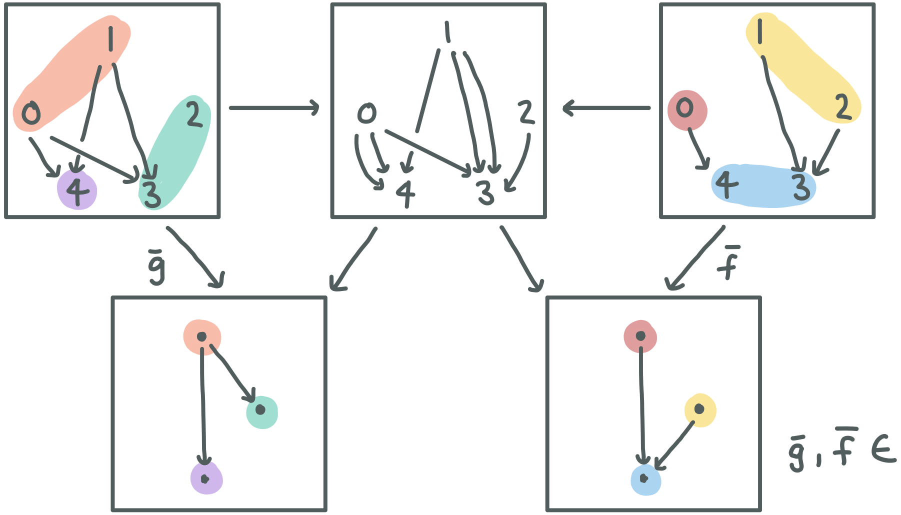
$\bar{g}, \bar{f} \in \mathcal{D}$

Main theorem: an adjoint triple

$$\mathrm{SymLens}(A,B) \xrightarrow[L]{\text{"pushout"}} \mathrm{SpnLens}(A,B)$$

$$\begin{tikzcd} & X^+ \arrow[dl, "\bar{g}"'] \arrow[dr, "f"] & \\ A & X_0 \arrow[u] \arrow[d] & B \\ & X^- \arrow[ul, "g"] \arrow[ur, "\bar{f}"'] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} X^+ \arrow[rr] \arrow[dr, "\bar{g}"'] & & \hat{X} \arrow[dl] \arrow[dr] & & X^- \arrow[ll] \arrow[dl, "\bar{f}"] \\ & A & & B & \end{tikzcd}$$

where $\hat{X} = X^+ \sqcup_{X_0} X^-$

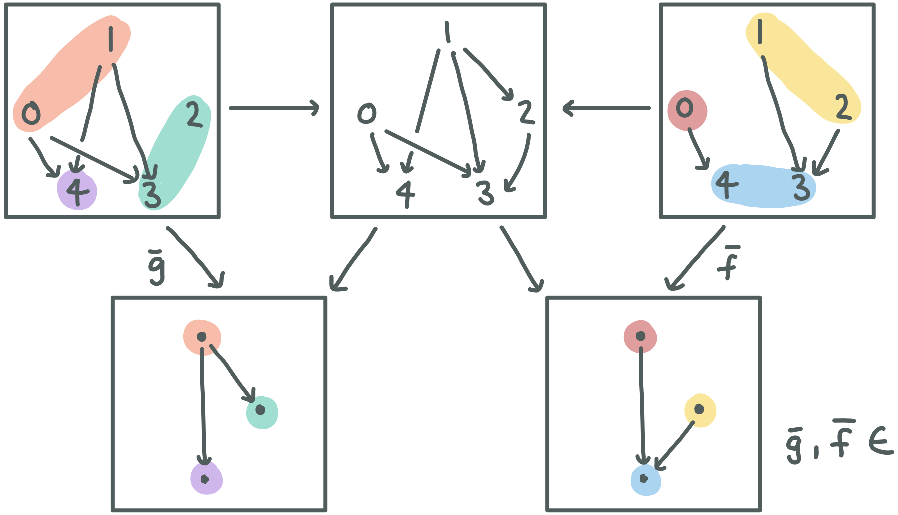


$\bar{g}, \bar{f} \in \mathcal{D}$

$$\mathrm{SymLens}(A,B) \xrightarrow[R]{\text{"factorise"}} \mathrm{SpnLens}(A,B)$$

$$\begin{tikzcd} & X^+ \arrow[dl, "\bar{g}"'] \arrow[dr, "f"] & \\ A & X_0 \arrow[l, "g_0"'] \arrow[r, "f_0"] & B \\ & X^- \arrow[ul, "g"] \arrow[ur, "\bar{f}"'] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} X^+ \arrow[rr] \arrow[dr, "\bar{g}"'] & & \tilde{X} \arrow[dl] \arrow[dr] & & X^- \arrow[ll] \arrow[dl, "\bar{f}"] \\ & A & & B & \end{tikzcd}$$

where $\tilde{X} = \mathrm{Im}\left(X_0 \xrightarrow{\langle g_0, f_0 \rangle} A \times B\right)$



$\bar{g}, \bar{f} \in \mathcal{D}$

Summary

GOAL 1: Develop a diagrammatic framework for lenses.

RESULT:

$$
\begin{tikzcd}
 & \Lambda \arrow[dl, "\varphi"'] \arrow[dr, "\bar{f}"] \arrow[d, phantom, "\circlearrowright"] & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
\qquad \varphi \in \mathcal{E} \quad \bar{f} \in \mathcal{D}
$$

Asymmetric lens

$$
\begin{tikzcd}
 & X^{+} \arrow[dl, "\bar{g}"'] \arrow[dr, "f"] & \\
A & X_{0} \arrow[l, "g_{0}"'] \arrow[r, "f_{0}"] \arrow[u] \arrow[d] & B \\
 & X^{-} \arrow[ul, "g"] \arrow[ur, "\bar{f}"'] &
\end{tikzcd}
\qquad \bar{g}, \bar{f} \in \mathcal{D}
$$

Symmetric lens

GOAL 2: Understand the relationship between symmetric & asymmetric lenses.

RESULT:

$$
\begin{tikzcd}
\mathrm{SymLens}(A,B) \arrow[r, hook, bend left, "L"] \arrow[r, hook, bend right, "R"'] & \mathrm{SpnLens}(A,B) \arrow[l, "M" description]
\end{tikzcd}
\qquad L \dashv M \dashv R
$$

where R is reflective & L is coreflective