Characterising split opfibrations using lenses

Bryce Clarke

Wednesday 4 December 2019

63rd Annual Meeting of the Australian Mathematical Society

CoACT, Macquarie University

bryce.clarke1@students.mq.edu.au

Motivation

Idea

Split opfibration = functor + lifting + universal property

- Split opfibrations are functors equipped with a suitable choice of opcartesian lifts.
- **Cat**-valued functors $\simeq$ split opfibrations
- ~~**Cat**($\mathcal{E}$)-valued internal functors~~ $\rightsquigarrow$ *internal* split opfibrations
- (internal) split opfibrations = algebras for a particular monad = functor + (lifting + universal property)
- **Problem**: these definitions appear to be quite different.
- **Solution**: characterise split opfibrations using delta lenses.
  Split opfibrations = (functor + lifting) + universal property

Background

Discrete opfibrations

Definition

A functor $G\colon \mathbf{S} \to \mathbf{V}$ is a discrete opfibration if for all objects S in $\mathbf{S}$ and morphisms $\alpha\colon GS \to V$ in $\mathbf{V}$, there exists a unique morphism $\widehat{\alpha}\colon S \to S'$ such that $G\widehat{\alpha} = \alpha$.

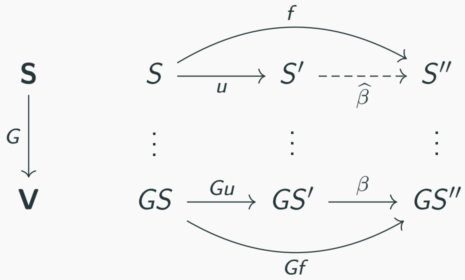
$$
\begin{array}{ccccc}
\mathbf{S} & \qquad & S & \overset{\widehat{\alpha}}{\dashrightarrow} & S' \\
{\scriptstyle G}\big\downarrow & & \vdots & & \vdots \\
\mathbf{V} & & GS & \xrightarrow{\ \alpha\ } & V
\end{array}
$$

Every functor $F\colon \mathbf{V} \to \mathbf{Set}$ yields a discrete opfibration $\int F \to \mathbf{V}$ via Grothendieck construction, and vice versa.

Opcartesian morphisms for a functor $G\colon \mathbf{S} \to \mathbf{V}$

Definition

A morphism $u\colon S \to S'$ in $\mathbf{S}$ is opcartesian if for all morphisms $f\colon S \to S''$ in $\mathbf{S}$ and $\beta\colon GS' \to GS''$ in $\mathbf{V}$ such that $\beta \circ Gu = Gf$, there exists a unique morphism $\widehat{\beta}\colon S' \to S''$ such that $G\widehat{\beta} = \beta$ and $\widehat{\beta} \circ u = f$.



Thus opcartesian morphisms satisfy a *universal property*.

Split opfibrations

Definition

A split opfibration is a pair (G, k) consisting of a functor $G\colon \mathbf{S} \to \mathbf{V}$ together with a function k, called a *splitting*,

$$(S, \alpha\colon GS \to V) \quad \longmapsto \quad k(S,\alpha)\colon S \to p(S,\alpha)$$

satisfying the following axioms:

(1) $Gk(S,\alpha) = \alpha$

(2) $k(S, 1_{GS}) = 1_S$

(3) $k(S, \beta \circ \alpha) = k(p(S,\alpha), \beta) \circ k(S,\alpha)$

(4) Each $k(S,\alpha)$ is opcartesian.

Every functor $F\colon \mathbf{V} \to \mathbf{Cat}$ yields a split opfibration $\int F \to \mathbf{V}$ via Grothendieck construction, and vice versa.

Characterising split opfibrations using lenses and décalage

Delta lenses

Definition (Diskin, Xiong, Czarnecki, 2011)

A delta lens is a pair (G, k) consisting of a functor $G\colon \mathbf{S} \to \mathbf{V}$ together with a function k, called a *lifting* (or *put*),

$$(S, \alpha\colon GS \to V) \quad \longmapsto \quad k(S, \alpha)\colon S \to p(S, \alpha)$$

satisfying the following axioms:

(1) $Gk(S, \alpha) = \alpha$

(2) $k(S, 1_{GS}) = 1_S$

(3) $k(S, \beta \circ \alpha) = k(p(S, \alpha), \beta) \circ k(S, \alpha)$

A split opfibration is a delta lens where each $k(S, \alpha)$ is opcartesian.

Characterising delta lenses using functors

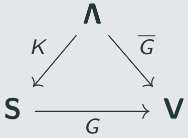
Lemma

Every delta lens $(G, k)\colon \mathbf{S} \to \mathbf{V}$ *induces a category* $\mathbf{\Lambda}$ *with the same objects as* $\mathbf{S}$ *and with morphisms given by formal pairs:*

$$S \xrightarrow{(S,\alpha)} p(S, \alpha)$$

Proposition

Every delta lens may be represented as a diagram in **Cat** *given by,*



where K *is identity-on-objects and* $\overline{G}$ *is a discrete opfibration.*

Décalage

Definition

There is a comonad $\mathsf{Dec_r}$ on **Cat** called the right décalage which assigns to each category **S** the sum of its slice categories:

$$\mathsf{Dec_r}(\mathbf{S}) = \sum_{X \in \mathbf{S}} \mathbf{S}/X$$

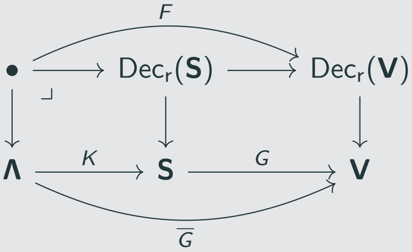
The counit components are *discrete fibrations* given by:

$$\begin{array}{ccc} S & \xrightarrow{f} & S' \\ & {}_{gf}\searrow \quad \swarrow_{g} & \\ & X & \end{array} \qquad \longmapsto \qquad S \xrightarrow{f} S'$$

Main Theorem

Theorem

A delta lens $(G, k)\colon \mathbf{S} \to \mathbf{V}$ *is a split opfibration if and only if the composite functor* F *shown below is a discrete opfibration.*



Split opfibration = delta lens + universal property

Characterising split opfibrations using lenses and double categories

Split opfibrations as algebras for a monad

Let $\mathbf{V}$ be a category. There is a monad M on the slice category $\mathbf{Cat} \,/\, \mathbf{V}$ whose action on objects is given by,

$$G \colon \mathbf{S} \to \mathbf{V} \quad \longmapsto \quad \Pi_{\mathbf{V}} \colon G \downarrow \mathbf{V} \to \mathbf{V}$$

where $\Pi_{\mathbf{V}}$ is the comma category projection.

Definition (Street, 1974)

A split opfibration is an algebra for the monad M.

Unpacking: split opfibration is a pair (G, P) consisting of functors $G \colon \mathbf{S} \to \mathbf{V}$ and $P \colon G \downarrow \mathbf{V} \to \mathbf{S}$ such that $GP = \Pi_{\mathbf{V}}$.

Split opfibration = functor + structure map

Generalises to any 2-category with pullbacks and comma objects.

Does this definition make sense?

Consider a split opfibration given by the pair (G, P), where the functor $P\colon G \downarrow \mathbf{V} \to \mathbf{S}$ has action on morphisms given by:

$$\begin{array}{ccc} GS & \xrightarrow{\alpha} & V \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle \beta} \\ GS' & \xrightarrow[\alpha']{} & V' \end{array} \quad \longmapsto \quad \begin{array}{c} P(S,\alpha) \\ \downarrow{\scriptstyle P\langle f,\beta\rangle} \\ P(S',\alpha') \end{array}$$

Proposition

Every split opfibration (G, P) yields a split opfibration of the form (G, k) where $k(S, \alpha) = P\langle 1_S, \alpha\rangle$ as depicted below:

$$\begin{array}{ccc} GS & \xrightarrow{1_{GS}} & GS \\ {\scriptstyle G(1_S)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ GS & \xrightarrow[\alpha]{} & V \end{array} \quad \longmapsto \quad \begin{array}{c} S \\ \downarrow{\scriptstyle P\langle 1_S,\alpha\rangle} \\ P(S,\alpha) \end{array}$$

Double categories

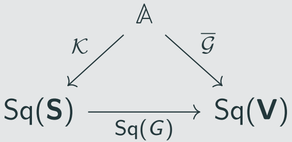
A double category is an internal category in **Cat**.

Lemma

Every split opfibration $(G, P)\colon \mathbf{S} \to \mathbf{V}$ *induces a double category* $\mathbb{A}$ *with domain and codomain maps given by* $\Pi_{\mathbf{S}}, P\colon G \downarrow \mathbf{V} \rightrightarrows \mathbf{S}$.

Proposition

Every split opfibration may be represented as a diagram in **Dbl** *given by,*



where $\mathcal{K}$ *is identity-on-objects and* $\overline{\mathcal{G}}$ *is a discrete opfibration.*

i.e. a lens *internal* to **Cat** between the double categories of squares.

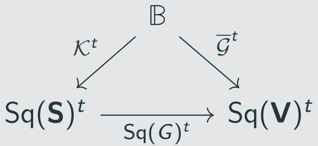
Connection between these characterisations

Lemma

Every split opfibration $(G, k)\colon \mathbf{S} \to \mathbf{V}$ *induces a double category* $\mathbb{B}$ *with domain and codomain maps given by* $\Pi_0, \Pi_1 \colon K \downarrow K \rightrightarrows \Lambda$.

Proposition

Every split opfibration $(G, k)\colon \mathbf{S} \to \mathbf{V}$ *may be represented as a diagram in* **Dbl** *given by:*



where $\mathbb{B}$ *is isomorphic to the transpose of* $\mathbb{A} = (G \downarrow \mathbf{V} \rightrightarrows \mathbf{S})$.

Summary

Summary of the talk

Key idea

Split opfibration = delta lens + universal property

- Delta lenses capture the *structure* of a split opfibration.
- Every split opfibration is a delta lens (G, k) where each lift $k(S, \alpha)$ has the *property* of being opcartesian.
- A delta lens is a split opfibration iff its representation in **Cat** satisfies a condition with respect to the right décalage.
- Every split opfibration may be represented as a diagram in **Dbl** – an *internal lens* between the double categories of squares.
- **Benefits**: Results generalise to $\mathbf{Cat}(\mathcal{E})$; new proofs of old results on split opfibrations; and clearer understanding.