AN INTRODUCTION TO ENRICHED COFUNCTORS

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ABSTRACT. Cofunctors are a kind of map between categories which lift morphisms along an object assignment. In this paper, we introduce cofunctors between categories enriched in a distributive monoidal category. We define a double category of enriched categories, enriched functors, and enriched cofunctors, whose horizontal and vertical 2-categories have 2-cells given by enriched natural transformations between functors and cofunctors, respectively. Enriched lenses are defined as a compatible enriched functor and enriched cofunctor pair; weighted lenses, which were introduced by Perrone, are precisely lenses enriched in weighted sets. Several other examples are also studied in detail.

1. INTRODUCTION

Cofunctors were introduced by Aguiar [Agu97] as a kind of morphism between categories, generalising the notion of *comorphism* between Lie groupoids [HM93]. Although cofunctors have not played a significant role in classical category theory, there has recently been a growing interest in their properties and applications. Ahman and Uustalu proved that polynomial comonads on $\mathcal{S}et$ are precisely small categories [AU16], whilst the comonad morphisms correspond to cofunctors [AU17]. The category $\mathcal{C}at^{\sharp}$ of small categories and cofunctors, and its relationship to polynomial functors, is the subject of ongoing work of Spivak and Nui [SN22]. Applications of cofunctors in mathematics and computer science have also appeared in a number of places [Gar19; Cla20b; Spi20; CG21; Gar21; Spi22].

The purpose of this paper is to generalise the notion of cofunctor to the setting of enriched category theory. One reason to do this is to provide an enriched counterpart to the theory of *internal cofunctors* developed by Aguiar [Agu97, Section 4.2]. The goal of both enriched and internal category theory is to not only reach a better understanding of (ordinary) category theory, but to also extend known concepts to a wider range of examples. The primary contributions of this paper are to establish the basic theory of cofunctors enriched over a *distributive monoidal category*, and to show how this captures a number of examples in the literature (see Section 6).

A key motivation for initiating the study of enriched cofunctors comes from recent work by Perrone on a variant of delta lenses called *weighted lenses* [Per21]. *Delta lenses* are a kind of morphism between categories [DXC11], generalising the notion of split opfibration [JR13]. A *weighted category* is a category in which every morphism is equipped with an extended non-negative real number called a *weight* [Gra07], and a weighted lens is

TABLE 1. Comparison of (enriched) functors and cofunctors.

	Object assignment	Morphism assignment
functor $F\colon \mathcal{A} \to \mathcal{B}$	$F\colon \operatorname{obj}(\mathcal{A}) \to \operatorname{obj}(\mathcal{B})$	$[F_{a,x}]\colon \displaystyle\sum_{x \in F^{-1}\{b\}} \mathcal{A}(a,x) \to \mathcal{B}(Fa, b)$
cofunctor $\Phi\colon \mathcal{A} \nrightarrow \mathcal{B}$	$\Phi\colon \operatorname{obj}(\mathcal{A}) \to \operatorname{obj}(\mathcal{B})$	$\Phi_{a,b}\colon \mathcal{B}(Fa, b) \to \displaystyle\sum_{x \in \Phi^{-1}\{b\}} \mathcal{A}(a,x)$

the same as a delta lens whose underlying functor and cofunctor are *weight non-increasing*. However, weighted categories are precisely categories enriched in weighted sets, so it is natural to wonder whether weighted lenses are examples of some notion of *enriched lens*. We answer this question affirmatively (see Definition 8 and Section 6.5).

Similar to Ahman and Usutalu's characterisation of delta lenses as consisting of a functor and cofunctor that are compatible with each other in a certain way [AU17], we define enriched lens in terms of an analogous notion of *compatibility* for enriched functors and cofunctors. To fully understand compatibility, it is important, first, to appreciate the duality between the concepts of enriched functor and cofunctor.

At first glance, functors and cofunctors appear to be quite different. A functor assigns objects and morphisms in the same direction, whereas a cofunctor lifts morphisms in the opposite direction to its assignment on objects. In addition, enriched functors may be enriched in *any* monoidal category, whilst our definition of enriched cofunctor uses the assumption that the base of enrichment is distributive monoidal in an essential way. There is, however, a hidden duality between functors and cofunctors (see Table 1).

For a functor $F\colon \mathcal{A} \to \mathcal{B}$, the data of the functions $F_{a,b}\colon \mathcal{A}(a,a') \to \mathcal{B}(Fa, Fa')$ is equivalent, by the universal property of the coproduct, to the function $[F_{a,x}]$ in Table 1. Cofunctors $\Phi\colon \mathcal{A} \nrightarrow \mathcal{B}$ are dual to functors in the sense that the direction of the corresponding morphism assignments are reversed. The compatibility between a cofunctor Φ and a functor F may then be expressed as the requirement that their morphism assignments, as depicted in Table 1, are a section-retraction pair. This duality extends to functors and cofunctors enriched in a distributive monoidal category.

It is also natural to wonder whether there are more general notions of "compatibility" between (enriched) functors and cofunctors, and whether there is a setting in which they may be studied together. One approach is to try to construct a double category of (enriched) categories, functors, and cofunctors, an idea first suggested by Paré (for the non-enriched setting) [Par20]. For (ordinary) functors and cofunctors, an appropriate choice of cell for such a double category is the notion of *compatible square*, and delta lenses are compatible squares of a certain form [Cla22; Di 21]. For enriched functors and

cofunctors, we define an analogous notion of compatible square, yielding the flat double category $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}_{\mathrm{flat}}$; enriched lenses are likewise certain cells therein.

It is unsatisfying that we cannot recover from $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}_{\mathrm{flat}}$ the notion of enriched natural transformation. Surprisingly, there is another double category $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}$, which has $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}_{\mathrm{flat}}$ as a double subcategory, whose horizontal 2-category is the 2-category of enriched categories, functors, and natural transformations. Furthermore, the 2-cells of the vertical 2-category of $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}$ are a notion of enriched natural transformation between enriched cofunctors analogous to Aguiar's *natural cotransformation* in the internal setting.

Outline of the paper. In Section 2, we recall the definition of the 2-category $\mathcal{V}\text{-}\mathcal{C}\mathrm{at}$ of categories, functors, and natural transformations enriched in a monoidal category $\mathcal{V}$.

In Section 3, we define the new notions of cofunctor enriched in a distributive monoidal category $\mathcal{V}$, and enriched natural transformation between them. Additionally, we construct a 2-category $\mathcal{V}\text{-}\mathcal{C}\mathrm{at}^{\sharp}$ of enriched categories, cofunctors, and natural transformations.

In Section 4, we define the double category $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}$ of enriched categories, functors, cofunctors, and natural transformations, and show how $\mathcal{V}\text{-}\mathcal{C}\mathrm{at}$ and $\mathcal{V}\text{-}\mathcal{C}\mathrm{at}^{\sharp}$ are, respectively, the horizontal and vertical 2-categories of this double category.

In Section 5, we introduce the notion of enriched lens. We also define a flat double subcategory $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}_{\mathrm{flat}}$ of $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}$ whose cells are compatible squares, and characterise enriched lenses as compatible squares of a certain form.

In Section 6, we describe the cofunctors and lenses enriched in
- the category of sets and functions equipped with the cartesian monoidal product;
- the category of vector spaces equipped with the usual tensor product;
- the thin category of extended non-negative real numbers equipped with addition as the monoidal product;
- the free coproduct completion of a monoidal category; and,
- the category of weighted sets.

Future outlook. The current version of this paper is the first part of a larger body of research currently under preparation. In a forthcoming version of this paper, we plan to
- explain the equivalence between enriched cofunctors and certain spans of enriched functors, which holds when the base of enrichment is extensive [CD22];
- introduce the theory of *monad retromorphisms* in a double category with a functorial choice of companions [Cla22]—in the double categories of matrices in a distributive monoidal category $\mathcal{V}$, of spans in a category $\mathcal{E}$ with pullbacks, and of comodules in a monoidal category $\mathcal{W}$ with nice equalisers, these are, respectively, cofunctors enriched in $\mathcal{V}$, cofunctors internal to $\mathcal{E}$ [Cla20a], and cofunctors internal to $\mathcal{W}$ [Agu97]; and,
- give a new perspective on labelled transition systems and bisimulation in terms of enrichment in the free quantale $P(L^*)$ on the free monoid L^* of the label set L.

2. Enriched categories, functors, and natural transformations

In this section, we recall from Kelly [Kel05] the concepts of enriched category, enriched functor, and enriched natural transformation over a fixed monoidal category $\mathcal{V}$. We also define the corresponding 2-category $\mathcal{V}$-$\mathcal{C}$at.

Definition 1. An *enriched category* $\mathcal{A}$ consists of a set $\mathrm{obj}(\mathcal{A})$ of objects, a hom-object $\mathcal{A}(x, y)$ in $\mathcal{V}$ for each pair of objects, an identity element $\eta_x \colon I \to \mathcal{A}(x, x)$ for each object, and a composition map $\mu_{x,y,z} \colon \mathcal{A}(x, y) \otimes \mathcal{A}(y, z) \to \mathcal{A}(x, z)$ for each triple of objects, such that the following diagrams, which encode the unitality and associativity laws, commute.

$$
\begin{array}{ccc}
I \otimes \mathcal{A}(x,y) & & \mathcal{A}(x,y) \otimes I \\
\eta_x \otimes \mathrm{id} \downarrow \quad \xrightarrow{\cong} & \xleftarrow{\cong} \quad & \downarrow \mathrm{id} \otimes \eta_y \\
\mathcal{A}(x,x) \otimes \mathcal{A}(x,y) \xrightarrow{\mu_{x,x,y}} \mathcal{A}(x,y) & \xleftarrow{\mu_{x,y,y}} & \mathcal{A}(x,y) \otimes \mathcal{A}(y,y)
\end{array}
$$

$$
\begin{array}{ccc}
\big(\mathcal{A}(w,x) \otimes \mathcal{A}(x,y)\big) \otimes \mathcal{A}(y,z) & \xrightarrow{\mu_{w,x,y} \otimes \mathrm{id}} & \mathcal{A}(w,y) \otimes \mathcal{A}(y,z) \\
\cong \downarrow & & \\
\mathcal{A}(w,x) \otimes \big(\mathcal{A}(x,y) \otimes \mathcal{A}(y,z)\big) & & \downarrow \mu_{w,y,z} \\
\mathrm{id} \otimes \mu_{x,y,z} \downarrow & & \\
\mathcal{A}(w,x) \otimes \mathcal{A}(x,z) & \xrightarrow{\mu_{w,x,z}} & \mathcal{A}(w,z)
\end{array}
$$

Definition 2. An *enriched functor* $F \colon \mathcal{A} \to \mathcal{B}$ consists of a function $F \colon \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$ and a family $F_{a,a'} \colon \mathcal{A}(a, a') \to \mathcal{B}(Fa, Fa')$ of morphisms in $\mathcal{V}$, such that the following diagrams, which encode preservation of identities and composites, commute.

$$
\begin{array}{ccc}
& I & \\
\eta_a \swarrow & & \searrow \eta_{Fa} \\
\mathcal{A}(a,a) \xrightarrow{F_{a,a}} & & \mathcal{B}(Fa, Fa)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{A}(a,a') \otimes \mathcal{A}(a',a'') & \xrightarrow{\mu_{a,a',a''}} & \mathcal{A}(a,a'') \\
F_{a,a'} \otimes F_{a',a''} \downarrow & & \downarrow F_{a,a''} \\
\mathcal{B}(Fa, Fa') \otimes \mathcal{B}(Fa', Fa'') & \xrightarrow{\mu_{Fa,Fa',Fa''}} & \mathcal{B}(Fa, Fa'')
\end{array}
$$

Definition 3. An *enriched natural transformation* $\tau \colon F \Rightarrow G \colon \mathcal{A} \to \mathcal{B}$ of enriched functors consists of a family $\tau_a \colon I \to \mathcal{B}(Fa, Ga)$ of morphisms in $\mathcal{V}$, such that the following diagram, which encodes naturality, commutes.

$$
\begin{array}{ccc}
I \otimes \mathcal{A}(a,a') \xleftarrow{\cong} \mathcal{A}(a,a') & \xrightarrow{\cong} & \mathcal{A}(a,a') \otimes I \\
\tau_a \otimes G_{a,a'} \downarrow & & \downarrow F_{a,a'} \otimes \tau_{a'} \\
\mathcal{B}(Fa, Ga) \otimes \mathcal{B}(Ga, Ga') \xrightarrow{\mu_{Fa,Ga,Ga'}} \mathcal{B}(Fa, Ga') & \xleftarrow{\mu_{Fa,Fa',Ga'}} & \mathcal{B}(Fa, Fa') \otimes \mathcal{B}(Fa', Ga')
\end{array}
$$

There is a 2-category $\mathcal{V}$-$\mathcal{C}$at whose objects are enriched categories, whose 1-cells are enriched functors, and whose 2-cells are enriched natural transformations.

3. Enriched cofunctors

In this section, we introduce the notions of enriched cofunctor and enriched natural transformation between enriched cofunctors over a fixed distributive monoidal category $\mathcal{V}$. We also define the corresponding 2-category $\mathcal{V}\text{-}\mathcal{C}\mathrm{at}^\sharp$.

Definition 4. An *enriched cofunctor* $\Phi\colon \mathcal{A} \nrightarrow \mathcal{B}$ consists of a function

$$\Phi\colon \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$$

called the *object map*, and a family

$$\Phi_{a,b}\colon \mathcal{B}(\Phi a, b) \longrightarrow \sum_{x \in X} \mathcal{A}(a, x) \qquad X = \Phi^{-1}\{b\}$$

of morphisms in $\mathcal{V}$, called *lifting maps*, such that the following diagrams, which encode preservation of identities and composites, commute.

$$
\begin{array}{ccc}
I & \xrightarrow{\;\eta_{\Phi a}\;} & \mathcal{B}(\Phi a, \Phi a) \\
{\scriptstyle \eta_a}\downarrow & & \downarrow{\scriptstyle \Phi_{a,\Phi a}} \\
\mathcal{A}(a, a) & \xrightarrow{\;\iota_a\;} & \sum_{x \in X} \mathcal{A}(a, x)
\end{array}
\qquad X = \Phi^{-1}\{\Phi a\}
$$

$$
\begin{array}{ccc}
\mathcal{B}(\Phi a, b) \otimes \mathcal{B}(b, b') & \xrightarrow{\;\mu_{\Phi a, b, b'}\;} & \mathcal{B}(\Phi a, b') \\
{\scriptstyle \Phi_{a,b}\otimes\mathrm{id}}\downarrow & & \\
\left(\sum_{x \in X} \mathcal{A}(a, x)\right) \otimes \mathcal{B}(b, b') & & \\
{\scriptstyle \cong}\downarrow & & \\
\sum_{x \in X} \mathcal{A}(a, x) \otimes \mathcal{B}(\Phi x, b') & & {\scriptstyle \Phi_{a,b'}} \\
{\scriptstyle \sum \mathrm{id}\otimes\Phi_{x,b'}}\downarrow & & \\
\sum_{x \in X} \mathcal{A}(a, x) \otimes \left(\sum_{y \in Y} \mathcal{A}(x, y)\right) & & \\
{\scriptstyle \cong}\downarrow & & \\
\sum_{y \in Y}\sum_{x \in X} \mathcal{A}(a, x) \otimes \mathcal{A}(x, y) & \xrightarrow{\;\sum[\mu_{a,x,y}]\;} & \sum_{y \in Y} \mathcal{A}(a, y)
\end{array}
\qquad
\begin{array}{l}
X = \Phi^{-1}\{b\} \\[4pt]
Y = \Phi^{-1}\{b'\}
\end{array}
$$

There are two surprises in the above definition—the distributivity requirement on the base of enrichment $\mathcal{V}$, and the indexing of the sum in the codomain of $\Phi_{a,b}$ by the fibre $\Phi^{-1}\{b\}$ rather than all of $\mathrm{obj}(\mathcal{A})$. Both of these are essential for formulating the axiom expressing the compatibility of the cofunctor with composition.

The composite of enriched cofunctors $\Phi\colon \mathcal{A} \nrightarrow \mathcal{B}$ and $\Psi\colon \mathcal{B} \nrightarrow \mathcal{C}$ has object map given by the composite of their object maps and lifting maps given by the composite morphism

$$\mathcal{C}(\Psi\Phi a, c) \xrightarrow{\Psi_{\Phi a, c}} \sum_{x \in X} \mathcal{B}(\Phi a, x) \xrightarrow{\sum \Phi_{a,x}} \sum_{x \in X} \sum_{y \in Y_x} \mathcal{A}(a, y) \xrightarrow{\cong} \sum_{z \in Z} \mathcal{A}(a, z).$$

$$X = \Psi^{-1}\{c\} \qquad Y_x = \Phi^{-1}\{x\} \qquad Z = (\Psi\Phi)^{-1}\{c\}$$

Whilst it is clear that composition of enriched cofunctors is strictly unital and also associative up to isomorphism; the proof that associativity is actually *strict* is more subtle.

Definition 5. An *enriched natural transformation* $\tau\colon \Phi \Rightarrow \Psi\colon \mathcal{A} \nrightarrow \mathcal{B}$ of enriched cofunctors consists of a family

$$\tau_a\colon I \longrightarrow \sum_{x \in X} \mathcal{A}(a, x) \qquad X = \Psi^{-1}\{\Phi a\}$$

of morphisms in $\mathcal{V}$, called the *components* of τ, such that the following diagram, which encodes naturality, commutes.

$$
\begin{array}{ccccc}
I \otimes \mathcal{B}(\Phi a, b) & \xleftarrow{\ \cong\ } \mathcal{B}(\Phi a, b) \xrightarrow{\ \cong\ } & \mathcal{B}(\Phi a, b) \otimes I \\
{\scriptstyle \tau_a \otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \Phi_{a,b} \otimes \mathrm{id}} \\
\left(\sum_{x \in X} \mathcal{A}(a, x)\right) \otimes \mathcal{B}(\Phi a, b) & & \left(\sum_{y \in Y} \mathcal{A}(a, y)\right) \otimes I \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
\sum_{x \in X} \mathcal{A}(a, x) \otimes \mathcal{B}(\Psi x, b) & & \sum_{y \in Y} \mathcal{A}(a, y) \otimes I \\
{\scriptstyle \sum \mathrm{id} \otimes \Psi_{x,b}}\downarrow & & \downarrow{\scriptstyle \sum \mathrm{id} \otimes \tau_y} \\
\sum_{x \in X} \mathcal{A}(a, x) \otimes \left(\sum_{z \in Z} \mathcal{A}(x, z)\right) & & \sum_{y \in Y} \mathcal{A}(a, y) \otimes \left(\sum_{z \in Z} \mathcal{A}(y, z)\right) \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
\sum_{z \in Z} \sum_{x \in X} \mathcal{A}(a, x) \otimes \mathcal{A}(x, z) \xrightarrow{\sum [\mu_{a,x,z}]} & \sum_{z \in Z} \mathcal{A}(a, z) \xleftarrow{\sum [\mu_{a,y,z}]} & \sum_{z \in Z} \sum_{y \in Y} \mathcal{A}(a, y) \otimes \mathcal{A}(y, z)
\end{array}
$$

$$X = \Psi^{-1}\{\Phi a\} \qquad Y = \Phi^{-1}\{b\} \qquad Z = \Psi^{-1}\{b\}$$

Remark. Aguiar uses the term *natural cotransformation* [Agu97] for what we simply refer to as *(enriched) natural transformation*. It will always be clear from context whether a given natural transformation is between a pair of functors or a pair of cofunctors.

There is a 2-category $\mathcal{V}\text{-}\mathcal{C}\mathrm{at}^{\sharp}$ whose objects are enriched categories, whose 1-cells are enriched cofunctors, and whose 2-cells are enriched natural transformations. To define the composition of 2-cells in this 2-category, it suffices to define the vertical composition of 2-cells and also the left and right whiskering of 2-cells with 1-cells.

The *left whiskering*

$$\mathcal{D} \xrightarrow{\ \Upsilon\ } \mathcal{A} \overset{\Phi}{\underset{\Psi}{\Longrightarrow{\tau}}} \mathcal{B}$$

in $\mathcal{V}\text{-}\mathcal{C}\mathrm{at}^{\sharp}$ has component at $d \in \mathrm{obj}(\mathcal{D})$ given by the composite morphism

$$I \xrightarrow{\ \tau_{\Upsilon d}\ } \sum_{x \in X} \mathcal{A}(\Upsilon d, x) \xrightarrow{\sum \Upsilon_{d,x}} \sum_{x \in X} \sum_{y \in Y_x} \mathcal{D}(d, y) \xrightarrow{\ \cong\ } \sum_{z \in Z} \mathcal{D}(d, z).$$

$$X = \Psi^{-1}\{\Phi \Upsilon d\} \qquad Y_x = \Upsilon^{-1}\{x\} \qquad Z = (\Psi \Upsilon)^{-1}\{\Phi \Upsilon d\}$$

The *right whiskering*

$$\mathcal{A} \overset{\Phi}{\underset{\Psi}{\Longrightarrow{\tau}}} \mathcal{B} \xrightarrow{\ \Omega\ } \mathcal{C}$$

in $\mathcal{V}\text{-}\mathcal{C}\mathrm{at}^{\sharp}$ has component at $a \in \mathrm{obj}(\mathcal{A})$ given by the composite morphism

$$I \xrightarrow{\ \tau_a\ } \sum_{x \in X} \mathcal{A}(a, x) \xrightarrow{[\iota_x]} \sum_{y \in Y} \mathcal{A}(a, y).$$

$$X = \Psi^{-1}\{\Phi a\} \qquad Y = (\Omega \Psi)^{-1}\{\Omega \Phi a\}$$

The *vertical composite*

$$\mathcal{A} \xrightarrow[\ \Psi\ ]{} \mathcal{B}$$

in $\mathcal{V}\text{-}\mathcal{C}\mathrm{at}^{\sharp}$ has component at $a \in \mathrm{obj}(\mathcal{A})$ given by the composite morphism

$$I \xrightarrow{\ \tau_a\ } \sum_{x \in X} \mathcal{A}(a, x) \xrightarrow{\ \cong\ } \sum_{x \in X} \mathcal{A}(a, x) \otimes I \xrightarrow{\sum \mathrm{id} \otimes \sigma_x}$$
$$\hookrightarrow \sum_{x \in X} \mathcal{A}(a, x) \otimes \Big(\sum_{y \in Y} \mathcal{A}(x, y) \Big) \xrightarrow{\ \cong\ } \sum_{x \in X} \mathcal{A}(a, x) \otimes \mathcal{A}(x, y) \xrightarrow{\sum [\mu_{a,x,y}]} \sum_{y \in Y} \mathcal{A}(a, y).$$

$$X = \Psi^{-1}\{\Phi a\} \qquad Y = \Upsilon^{-1}\{\Phi a\}$$

4. The double category of enriched categories, functors, and cofunctors

In this section, we introduce the double category $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}$ whose objects are categories, whose horizontal morphisms are functors, and whose vertical morphisms are cofunctors, all enriched over a fixed distributive monoidal category $\mathcal{V}$. We reuse the name *enriched natural transformation* for its cells because this notion subsumes both previously-defined notions of enriched natural transformation.

Definition 6. A *enriched natural transformation*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle\Phi}\downarrow & \overset{\tau}{\Swarrow} & \downarrow{\scriptstyle\Psi} \\ \mathcal{B} & \xrightarrow{G} & \mathcal{D} \end{array}$$

of enriched functors and cofunctors consists of a family of morphisms in $\mathcal{V}$

$$\tau_a \colon I \longrightarrow \sum_{x \in X} \mathcal{C}(Fa, x) \qquad X = \Psi^{-1}\{G\Phi a\}$$

called the *components* of τ, such that the following diagram, encoding naturality, commutes.

$$\begin{array}{ccc}
I \otimes \mathcal{B}(\Phi a, b) \xleftarrow{\;\;\cong\;\;} \mathcal{B}(\Phi a, b) \xrightarrow{\;\;\cong\;\;} \mathcal{B}(\Phi a, b) \otimes I \\
{\scriptstyle \tau_a \otimes G_{\Phi a,b}}\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \Phi_{a,b}\otimes \mathrm{id}} \\
\Big(\sum_{x\in X} \mathcal{C}(Fa,x)\Big) \otimes \mathcal{D}(G\Phi a, Gb) \qquad\qquad \Big(\sum_{y\in Y} \mathcal{A}(a,y)\Big)\otimes I \\
{\scriptstyle\cong}\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle\cong} \\
\sum_{x\in X}\mathcal{C}(Fa,x)\otimes \mathcal{D}(\Psi x, Gb) \qquad\qquad \sum_{y\in Y}\mathcal{A}(a,y)\otimes I \\
{\scriptstyle\sum \mathrm{id}\otimes\Psi_{x,Gb}}\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle\sum F_{a,y}\otimes\tau_y} \\
\sum_{x\in X}\mathcal{C}(Fa,x)\otimes\Big(\sum_{z\in Z}\mathcal{C}(x,z)\Big) \qquad \sum_{y\in Y}\mathcal{C}(Fa,Fy)\otimes\Big(\sum_{z\in Z}\mathcal{C}(Fy,z)\Big) \\
{\scriptstyle\cong}\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle\cong} \\
\sum_{z\in Z}\sum_{x\in X}\mathcal{C}(Fa,x)\otimes\mathcal{C}(x,z) \xrightarrow{\sum[\mu_{Fa,x,z}]} \sum_{z\in Z}\mathcal{C}(Fa,z) \xleftarrow{\sum[\mu_{Fa,Fy,z}]} \sum_{z\in Z}\sum_{y\in Y}\mathcal{C}(Fa,Fy)\otimes\mathcal{C}(Fy,z)
\end{array}$$

$$X = \Psi^{-1}\{G\Phi a\} \qquad Y = \Phi^{-1}\{b\} \qquad Z = \Psi^{-1}\{Gb\}$$

The *horizontal composite*

$$\begin{array}{ccccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{C} & \xrightarrow{H} & \mathcal{E} \\
{\scriptstyle\Phi}\downarrow & \overset{\tau}{\Swarrow} & \downarrow{\scriptstyle\Psi} & \overset{\sigma}{\Swarrow} & \downarrow{\scriptstyle\Upsilon} \\
\mathcal{B} & \xrightarrow{G} & \mathcal{D} & \xrightarrow{K} & \mathcal{F}
\end{array}$$

in $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}$ has component at $a \in \mathrm{obj}(\mathcal{A})$ given by the composite morphism

$$I \xrightarrow{\tau_a} \sum_{x\in X}\mathcal{C}(Fa,x) \xrightarrow{\sum H_{Fa,x}} \sum_{x\in X}\mathcal{E}(HFa,Hx) \xrightarrow{\cong} \sum_{x\in X}\mathcal{E}(HFa,Hx)\otimes I \xrightarrow{\sum \mathrm{id}\otimes\sigma_x}$$

$$\sum_{y\in Y}\sum_{x\in X}\mathcal{E}(HFa,Hx)\otimes\mathcal{E}(Hx,y) \xleftarrow{\cong} \sum_{x\in X}\mathcal{E}(HFa,Hx)\otimes\Big(\sum_{y\in Y}\mathcal{E}(Hx,y)\Big) \longleftarrow$$

$$\xrightarrow{\sum[\mu_{HFa,Hx,y}]} \sum_{y\in Y}\mathcal{E}(HFa,y).$$

$$X = \Psi^{-1}\{G\Phi a\} \qquad Y = \Upsilon^{-1}\{KG\Phi a\}$$

The *vertical composite*

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\ F\ } & \mathcal{C} \\
\Phi \downarrow & \Downarrow^{\tau} & \downarrow \Psi \\
\mathcal{B} & \xrightarrow{\ G\ } & \mathcal{D} \\
\Upsilon \downarrow & \Downarrow^{\sigma} & \downarrow \Omega \\
\mathcal{E} & \xrightarrow{\ H\ } & \mathcal{F}
\end{array}
$$

in $\mathcal{V}$-$\mathbb{C}\mathrm{of}$ has component at $a \in \mathrm{obj}(\mathcal{A})$ given by the composite morphism

$$
I \xrightarrow{\tau_a} \sum_{x \in X} \mathcal{C}(Fa, x) \xrightarrow{\cong} \sum_{x \in X} \mathcal{C}(Fa, x) \otimes I \xrightarrow{\sigma_{\Phi a}} \sum_{x \in X} \mathcal{C}(Fa, x) \otimes \Big(\sum_{y \in Y} \mathcal{D}(G\Phi a, y) \Big) \xrightarrow{\cong}
$$

$$
\sum_{y \in Y} \sum_{x \in X} \mathcal{C}(Fa, x) \otimes \Big(\sum_{z \in Z_y} \mathcal{C}(x, z) \Big) \xleftarrow{\sum \sum \mathrm{id} \otimes \Psi_{x,y}} \sum_{y \in Y} \sum_{x \in X} \mathcal{C}(Fa, x) \otimes \mathcal{D}(\Psi x, y) \leftarrow
$$

$$
\xrightarrow{\cong} \sum_{w \in W} \sum_{x \in X} \mathcal{C}(Fa, x) \otimes \mathcal{C}(x, w) \xrightarrow{\sum [\mu_{Fa,x,w}]} \sum_{w \in W} \mathcal{C}(Fa, w).
$$

$$
X = \Psi^{-1}\{G\Phi a\} \qquad Y = \Omega^{-1}\{H\Upsilon \Phi a\} \qquad Z_y = \Psi^{-1}\{y\} \qquad W = (\Omega\Psi)^{-1}\{H\Upsilon\Phi a\}
$$

Recall that every double category has a horizontal and a vertical 2-category [GP99].

Proposition 7. *The horizontal and vertical* 2-*categories of* $\mathcal{V}$-$\mathbb{C}\mathrm{of}$ *are* $\mathcal{V}$-$\mathcal{C}\mathrm{at}$ *and* $\mathcal{V}$-$\mathcal{C}\mathrm{at}^{\sharp}$.

5. Enriched lenses as compatible squares

In this section, we introduce the notion of enriched lens as an enriched functor and enriched cofunctor that satisfy a certain compatibility condition. We also introduce the flat double subcategory $\mathcal{V}$-$\mathbb{C}\mathrm{of}_{\mathrm{flat}}$ of $\mathcal{V}$-$\mathbb{C}\mathrm{of}$, whose cells are called *compatible squares*, and show that enriched lenses arise naturally as certain cells in this double category.

Definition 8. An *enriched lens* $(F, \Phi)\colon \mathcal{A} \rightleftharpoons \mathcal{B}$ consists of an enriched functor $F\colon \mathcal{A} \to \mathcal{B}$ and an enriched cofunctor $\Phi\colon \mathcal{A} \nrightarrow \mathcal{B}$, such that $Fa = \Phi a$ for all $a \in \mathrm{obj}(\mathcal{A})$ and the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{B}(Fa, b) & \xrightarrow{\quad \mathrm{id} \quad} & \mathcal{B}(Fa, b) \\
{}_{\Phi_{a,b}}\searrow & & \nearrow_{[F_{a,x}]} \\
& \sum_{x \in X} \mathcal{A}(a, x) &
\end{array}
\qquad X = F^{-1}\{b\} \qquad (1)
$$

There is a category $\mathcal{V}$-$\mathcal{L}\mathrm{ens}$ whose objects are enriched categories and whose morphisms are enriched lenses, where composition of enriched lenses is defined by the composition of the underlying enriched functors and enriched cofunctors.

Remark. The category $\mathcal{V}$-$\mathcal{L}\mathrm{ens}$ also has a 2-category structure. A 2-cell between a pair of enriched lenses consists of a enriched natural transformation between their underlying

functors and an enriched natural transformation between their underlying cofunctors that are compatible with each other (in a similar sense to Definition 9 below).

Definition 9. A square of functors and cofunctors

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{C} \\
{\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Psi} \\
\mathcal{B} & \xrightarrow{G} & \mathcal{D}
\end{array}
\tag{2}
$$

is a *compatible square* if $G\Phi a = \Psi F a$, and the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{B}(\Phi a, b) & \xrightarrow{G_{\Phi a, b}} & \mathcal{D}(\Psi F a, G b) \\
{\scriptstyle\Phi_{a,b}}\downarrow & & \downarrow{\scriptstyle\Psi_{Fa,Gb}} \\
\displaystyle\sum_{x\in X}\mathcal{A}(a,x) & \xrightarrow{\sum F_{a,x}} & \displaystyle\sum_{y\in Y}\mathcal{C}(Fa,y)
\end{array}
\qquad
\begin{array}{l}
X = \Phi^{-1}\{b\} \\
Y = \Psi^{-1}\{Gb\}
\end{array}
$$

Remark. We should think of compatibility as a commutativity-like notion for diagrams involving both functors and cofunctors of this particular shape.

There is a double category $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}_{\mathrm{flat}}$ whose objects are enriched categories, whose horizontal morphisms are enriched functors, whose vertical morphisms are enriched cofunctors, and whose cells are compatible squares. This is a double subcategory of $\mathcal{V}\text{-}\mathbb{C}\mathrm{of}$ via the double functor that sends a compatible square (2) to the enriched natural transformation with the same boundary whose component at $a \in \mathrm{obj}(\mathcal{A})$ is given by the composite below.

$$
I \xrightarrow{\eta_{Fa}} \mathcal{C}(Fa, Fa) \xrightarrow{\iota_{Fa}} \sum_{x\in X}\mathcal{C}(Fa, x) \qquad X = \Psi^{-1}\{G\Phi a\}
$$

Proposition 10. *An enriched lens* $(F, \Phi)\colon \mathcal{A} \rightleftharpoons \mathcal{B}$ *is precisely a compatible square*

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
{\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle 1_{\mathcal{B}}} \\
\mathcal{B} & \xrightarrow{1_{\mathcal{B}}} & \mathcal{B}
\end{array}
\;.
$$

6. Examples of enriched cofunctors and lenses

In this section, we describe the enriched cofunctors and lenses for several interesting bases of enrichment.

6.1. **Locally small categories.** Consider the category $\mathbb{S}\mathrm{et}$ equipped with the cartesian monoidal product. This is a distributive monoidal category, with coproducts given by disjoint union. Categories, cofunctors, and lenses enriched in $\mathbb{S}\mathrm{et}$ are precisely *locally small categories*, *cofunctors* [Agu97], and *delta lenses* [DXC11; JR13], respectively.

A cofunctor $\Phi\colon \mathcal{A} \nrightarrow \mathcal{B}$ consists of a function $\Phi\colon \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$ and a family

$$
\Phi_{a,b}\colon \mathcal{B}(\Phi a, b) \longrightarrow \sum_{x\in\Phi^{-1}\{b\}}\mathcal{A}(a,x)
\tag{3}
$$

of lifting maps such that $\Phi_{a,\Phi a}(1_{\Phi a}) = 1_a$ and $\Phi_{a,b'}(v \circ u) = \Phi_{a',b'}(v) \circ \Phi_{a,b}(u)$. The equation $\Phi(\operatorname{cod}\Phi_{a,b}(u)) = \operatorname{cod} u = b$ holds by construction. Examples of cofunctors $\mathcal{A} \nrightarrow \mathcal{B}$ include bijective-on-objects functors $\mathcal{B} \to \mathcal{A}$ and discrete opfibrations $\mathcal{A} \to \mathcal{B}$.

A delta lens $(F, \Phi) \colon \mathcal{A} \rightleftharpoons \mathcal{B}$ consists of a functor $F \colon \mathcal{A} \to \mathcal{B}$ and a cofunctor $\Phi \colon \mathcal{A} \nrightarrow \mathcal{B}$ such that $Fa = \Phi a$, and $F_{a,a'}\Phi_{a,b}(u) = u$ where $a' = \operatorname{cod}\Phi_{a,b}(u)$. Split opfibrations are precisely delta lenses (F, Φ) where each morphism $\Phi_{a,b}(u)$ is an opcartesian lift along F.

A natural transformation

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\ F\ } & \mathcal{C} \\
{\scriptstyle\Phi}\Big\updownarrow & \overset{\tau}{\searrow} & \Big\downarrow{\scriptstyle\Psi} \\
\mathcal{B} & \xrightarrow[\ G\ ]{} & \mathcal{D}
\end{array}
$$

of functors and cofunctors is a family

$$
\tau_a \colon Fa \to \operatorname{cod}\tau_a \qquad\qquad \Psi \operatorname{cod}\tau_a = G\Phi a
$$

of morphisms of $\mathcal{C}$ such that, for each $u \colon b \to b'$ in $\mathcal{B}$ and each $a \in \Phi^{-1}\{b\}$, the square

$$
\begin{array}{ccc}
Fa & \xrightarrow{\ \tau_a\ } & c \\
{\scriptstyle F\Phi_{a,b'}(u)}\Big\downarrow & & \Big\downarrow{\scriptstyle \Psi_{b,Gb'}G(u)} \\
Fa' & \xrightarrow[\ \tau_{a'}\ ]{} & c'
\end{array}
$$

$$
a' = \operatorname{cod}\Phi_{a,b'}(u) \qquad c = \operatorname{cod}\tau_a \qquad c' = \operatorname{cod}\tau_{a'} = \operatorname{cod}\Psi_{b,Gb'}G(u)
$$

in $\mathcal{C}$, called the *naturality square* of τ at u, always commutes.

6.2. **Linear categories.** Consider the category $\mathcal{V}ec_K$ of vector spaces over the field K, equipped with the usual tensor product of vector spaces. This is a distributive monoidal category, with coproducts given by the direct sum of vector spaces.

A category enriched in $\mathcal{V}ec_K$ is called a *linear category* or *algebroid* (an algebroid with one object is an algebra over a field). Aguiar showed that every category enriched in $\mathcal{V}ec_K$ may be viewed as a category internal to $\mathcal{V}ec_K$ (using a more general definition of category internal to a monoidal category with nice equalisers) [Agu97, Section 9.1]; we now observe that the internal cofunctors between a pair of enriched categories, via this identification, are exactly the enriched cofunctors between these enriched categories.

A cofunctor enriched in $\mathcal{V}ec_K$ will be called a *linear cofunctor*. Recall that the direct sum of a family of vector spaces is the subspace of their direct product formed by those tuples that are non-zero in only finitely many coordinates. Hence, a linear cofunctor $\Phi \colon \mathcal{A} \nrightarrow \mathcal{B}$ chooses, for each morphism $u \colon \Phi a \to b$ in $\mathcal{B}$, not just one lift, but a family

$$
\left\{ \Phi_{a,b}(u)_x \colon a \to x \right\}_{x \in X_{a,b,u}}
$$

of lifts, indexed by a finite subset $X_{a,b,u}$ of $\Phi^{-1}\{b\}$. Of course, these choices are subject to the requirement that the lifting maps $\Phi_{a,b}$ be linear. As finite direct sums are also

categorical products, when the fibre $\Phi^{-1}\{b\}$ is finite, the data of the lifting map

$$\Phi_{a,b}\colon \mathcal{B}(\Phi a, b) \longrightarrow \bigoplus_{x \in \Phi^{-1}\{b\}} \mathcal{A}(a, x)$$

is equivalent to the data of a family of linear maps

$$\left\{\Phi_{a,b,x}\colon \mathcal{B}(\Phi a, b) \longrightarrow \mathcal{A}(a, x)\right\}_{x \in \Phi^{-1}\{b\}}.$$

Recall that linear categories and functors may be viewed as (ordinary) categories and functors via base change along the forgetful functor $\mathcal{V}ec_K \to \mathcal{S}et$; this is possible because the forgetful functor is lax monoidal. As the forgetful functor does not preserve coproducts, the same is not true of linear cofunctors. In contrast, the free functor $\mathcal{S}et \to \mathcal{V}ec_K$ is both strong monoidal *and* preserves coproducts, so base change along the free functor works not only for categories and functors, but also for cofunctors.

6.3. **Lawvere metric spaces.** Consider the thin category $([0, \infty], \geqslant)$ of extended non-negative real numbers, equipped with addition $+\colon [0, \infty] \times [0, \infty] \to [0, \infty]$ as its monoidal product and the terminal object 0 as its monoidal unit. This is a distributive monoidal category, with coproducts given by infima (greatest lower bounds).

A category $\mathcal{A}$ enriched in $([0, \infty], \geqslant)$ is a *Lawvere metric space* [Law02]. It consists of a set of objects $\mathrm{obj}(\mathcal{A})$ together with a real number $\mathrm{d}(a, a') \in [0, \infty]$ for each pair of objects, which we think of as the *distance* from a to a'. The existence of the identity elements and composition maps of $\mathcal{A}$ is equivalent to requiring that

$$\mathrm{d}(a, a) = 0 \qquad \text{and} \qquad \mathrm{d}(a, a') + \mathrm{d}(a', a'') \geqslant \mathrm{d}(a, a'').$$

An functor $F\colon \mathcal{A} \to \mathcal{B}$ enriched in $([0, \infty], \geqslant)$ is a function $F\colon \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$ with

$$\mathrm{d}(a, a') \geqslant \mathrm{d}(Fa, Fa').$$

Such functions are called *non-expansive* or *1-Lipschitz*.

A cofunctor $\Phi\colon \mathcal{A} \nrightarrow \mathcal{B}$ enriched in $([0, \infty], \geqslant)$ is a function $\Phi\colon \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$ with

$$\mathrm{d}(\Phi a, b) \geqslant \inf_{x \in \Phi^{-1}\{b\}} \mathrm{d}(a, x). \tag{4}$$

This notion is closely related to the notion of *submetry*.[1]

A *submetry* $\Phi\colon \mathcal{A} \to \mathcal{B}$ between Lawvere metric spaces is a function $\Phi\colon \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$ such that for each $a \in \mathrm{obj}(\mathcal{A})$ and each $b \in \mathrm{obj}(\mathcal{B})$, there is an $x \in \Phi^{-1}\{b\}$ such that $d(\Phi a, b) \geqslant d(a, x)$. Although every submetry is an enriched cofunctor, not every enriched cofunctor is a submetry. For example, consider the Lawvere metric spaces $\mathcal{A}$ and $\mathcal{B}$ where

$$\mathrm{obj}(\mathcal{A}) = \{0\} \cup (1, 2], \qquad \mathrm{obj}(\mathcal{B}) = \{0\} \cup \{1\}, \quad \text{and} \quad d(x, y) = \begin{cases} y - x & \text{if } x \leqslant y, \\ \infty & \text{otherwise.} \end{cases}$$

---

[1]Many thanks to Callum Reader for pointing us towards this notion.

The enriched cofunctor $\Phi\colon \mathcal{A} \nrightarrow \mathcal{B}$ with object map

$$\Phi(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{otherwise} \end{cases}$$

is not a submetry because $d(\Phi(0), 1) = 1$ whilst $d(0, x) = x > 1$ for each $x \in \Phi^{-1}\{1\}$.

An enriched lens $F\colon \mathcal{A} \rightleftharpoons \mathcal{B}$ is a function $F\colon \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$ such that

$$d(Fa, b) = \inf_{x \in F^{-1}\{b\}} d(a, x).$$

Even an enriched lens is not necessarily a submetry, indeed the enriched cofunctor Φ defined in the example above is actually an enriched lens.

Remark. If a cofunctor $\Phi\colon \mathcal{A} \to \mathcal{B}$ enriched in $([0, \infty], \geqslant)$ has *finite* fibres, then the infimum in (4) is always attained, and so Φ is actually a submetry. Finding conditions on a cofunctor that are both sufficient and *necessary* to make it a submetry is an open problem.

6.4. **Enrichment in the free coproduct completion of a monoidal category.** One aspect where enriched cofunctors differ from enriched functors is the requirement for $\mathcal{V}$ to be a *distributive monoidal category*. Whilst it is not uncommon for many monoidal categories to be distributive, one might ask if it is possible to describe enriched cofunctors starting with an arbitrary monoidal category $(\mathcal{W}, \otimes, I)$.

The *free coproduct completion* $\mathcal{F}\mathrm{am}(\mathcal{C})$ of a category $\mathcal{C}$ is the comma category below.

$$\begin{array}{ccc} \mathcal{F}\mathrm{am}(\mathcal{C}) & \longrightarrow & * \\ \downarrow & \Rightarrow & \downarrow {\scriptstyle \mathcal{C}} \\ \mathcal{S}\mathrm{et} & \xhookrightarrow[\mathrm{disc}]{} & \mathcal{C}\mathrm{at} \end{array}$$

In detail, the objects of $\mathcal{F}\mathrm{am}(\mathcal{C})$ are *families* of objects in $\mathcal{C}$, that is, functions $J \to \mathrm{obj}(\mathcal{C})$ or, equivalently, functors $\mathrm{disc}(J) \to \mathcal{C}$ where $\mathrm{disc}(J)$ is the discrete category on the set J. A morphism in $\mathcal{F}\mathrm{am}(\mathcal{C})$ from $j\colon \mathrm{disc}(J) \to \mathcal{C}$ to $k\colon \mathrm{disc}(K) \to \mathcal{C}$ consists of a function $f\colon J \to K$ together with a natural transformation as depicted below.

$$\begin{array}{ccc} \mathrm{disc}(J) & \xrightarrow{\mathrm{disc}(f)} & \mathrm{disc}(K) \\ & {\scriptstyle j} \searrow \ \Rightarrow \ \swarrow {\scriptstyle k} & \\ & \mathcal{C} & \end{array}$$

The coproduct of objects $j\colon \mathrm{disc}(J) \to \mathcal{C}$ and $k\colon \mathrm{disc}(K) \to \mathcal{C}$ is given by

$$\mathrm{disc}(J \sqcup K) \cong \mathrm{disc}(J) + \mathrm{disc}(K) \xrightarrow{[j,k]} \mathcal{C}$$

and the initial object is given by the unique functor from the empty category to $\mathcal{C}$.

For a monoidal category $(\mathcal{W}, \otimes, I)$, the category $\mathcal{F}\mathrm{am}(\mathcal{W})$ is a distributive monoidal. The monoidal product of objects $j\colon \mathrm{disc}(J) \to \mathcal{C}$ and $k\colon \mathrm{disc}(K) \to \mathcal{C}$ is given by

$$\mathrm{disc}(J \times K) \cong \mathrm{disc}(J) \times \mathrm{disc}(K) \xrightarrow{j \times k} \mathcal{W} \times \mathcal{W} \xrightarrow{\otimes} \mathcal{W}$$

with unit given by the functor from the terminal category to $\mathcal{W}$ that chooses the object I. The distributivity of $\mathcal{F}\mathrm{am}(\mathcal{W})$ follows from the distributivity of $\mathcal{S}\mathrm{et}$ equipped with the cartesian monoidal product.

For simplicity, we consider the case where $\mathcal{W}$ is a thin category. We will write $x \geqslant y$ if and only if a morphism $x \to y$ exists.

A category enriched in $\mathcal{F}\mathrm{am}(\mathcal{W})$ consists of a locally small category $\mathcal{A}$ together with an object $|u|$ in $\mathcal{W}$ for each morphism $u\colon a \to a'$ in $\mathcal{A}$, such that

$$I \geqslant |1_a| \qquad \text{and} \qquad |u| \otimes |v| \geqslant |v \circ u|. \tag{5}$$

A functor enriched in $\mathcal{F}\mathrm{am}(\mathcal{W})$ is a functor $F\colon \mathcal{A} \to \mathcal{B}$ such that

$$|w| \geqslant |F_{a,a'}(w)|. \tag{6}$$

A cofunctor enriched in $\mathcal{F}\mathrm{am}(\mathcal{W})$ is a cofunctor $\varPhi\colon \mathcal{A} \nrightarrow \mathcal{B}$ such that

$$|u| \geqslant |\varPhi_{a,b}(u)|. \tag{7}$$

A lens enriched in $\mathcal{F}\mathrm{am}(\mathcal{W})$ is delta lens $(F, \varPhi)\colon \mathcal{A} \rightleftharpoons \mathcal{B}$ such that both (6) and (7) hold; by (1), it follows that $|u| = |\varPhi_{a,b}(u)|$ since $\mathcal{W}$ is a thin category.

6.5. **Weighted categories.** Let $\mathrm{w}\mathcal{S}\mathrm{et}$ be the free coproduct completion $\mathcal{F}\mathrm{am}([0, \infty], \geqslant)$ of the poset $([0, \infty], \geqslant)$ of extended non-negative real numbers (see Sections 6.3 and 6.4). The objects of this category are called *weighted sets* [Gra07]. Perrone introduced the notion of *weighted lenses* [Per21], which we now recognise as lenses enriched in $\mathrm{w}\mathcal{S}\mathrm{et}$.

A category enriched in $\mathrm{w}\mathcal{S}\mathrm{et}$ consists a locally small category $\mathcal{A}$ together with a *weight* $|u| \in [0, \infty]$ for each morphism $u\colon a \to a'$ in $\mathcal{A}$, such that the inequalities (5) hold. As the monoidal unit 0 is also a terminal object, the first inequality in (5) may be strengthened to the equality $|1_a| = 0$.

Weighted functors are functors such that the condition (6) holds. Similarly, weighted cofunctors are cofunctors such that the condition (7) holds. Weighted lenses are delta lenses such that the conditions (6) and (7) hold. Since $[0, \infty]$ is a poset, we may deduce the stronger condition $|\varPhi_{a,b}(u)| = |u|$ on the weight of a lift.

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References

[Agu97]   Marcelo Aguiar. "Internal categories and quantum groups". PhD thesis. Cornell University, 1997. URL: http://pi.math.cornell.edu/~maguiar/thesis2.pdf.

[AU16]   Danel Ahman and Tarmo Uustalu. "Directed Containers as Categories". In: *Electronic Proceedings in Theoretical Computer Science* 207 (2016). DOI: 10.4204/EPTCS.207.5.

[AU17]   Danel Ahman and Tarmo Uustalu. "Taking Updates Seriously". In: *CEUR Workshop Proceedings* 1827 (2017). URL: http://ceur-ws.org/Vol-1827/paper11.pdf.

[Cla20a]   Bryce Clarke. "Internal lenses as functors and cofunctors". In: *Electronic Proceedings in Theoretical Computer Science* 323 (2020). DOI: 10.4204/EPTCS.323.13.

[Cla20b]   Bryce Clarke. "Internal split opfibrations and cofunctors". In: *Theory and Applications of Categories* 35 (2020). URL: www.tac.mta.ca/tac/volumes/35/44/35-44abs.html.

[Cla22]   Bryce Clarke. "The double category of lenses". PhD thesis. Macquarie University, 2022. URL: https://bryceclarke.github.io/The_Double_Category_Of_Lenses_Phd_Thesis.pdf.

[CD22]   Bryce Clarke and Matthew Di Meglio. "Enriched lenses (extended abstract)". Applied Category Theory 2022 conference. 2022. URL: https://msp.cis.strath.ac.uk/act2022/papers/ACT2022_paper_4434.pdf.

[CG21]   Robin Cockett and Richard Garner. "Generalising the étale groupoid–complete pseudogroup correspondence". In: *Advances in Mathematics* 392 (2021). DOI: 10.1016/j.aim.2021.108030.

[Di 21]   Matthew Di Meglio. "The category of asymmetric lenses and its proxy pullbacks". Master's thesis. Macquarie University, 2021. DOI: 10.25949/20236449.v1.

[DXC11]   Zinovy Diskin, Yingfei Xiong, and Krzysztof Czarnecki. "From State- to Delta-Based Bidirectional Model Transformations: the Asymmetric Case". In: *Journal of Object Technology* 10 (2011). DOI: 10.5381/jot.2011.10.1.a6.

[Gar19]   Richard Garner. *Inner automorphisms of groupoids*. 2019. arXiv: 1907.10378.

[Gar21]   Richard Garner. "The costructure–cosemantics adjunction for comodels for computational effects". In: *Mathematical Structures in Computer Science* (2021). DOI: 10.1017/S0960129521000219.

[Gra07]   Marco Grandis. "Categories, norms, and weights". In: *Journal of Homotopy and Related Structures* 2 (2007). arXiv: math/0603298.

[GP99]   Marco Grandis and Robert Paré. "Limits in double categories". In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 40 (1999). URL: http://www.numdam.org/item/CTGDC_1999__40_3_162_0/.

[HM93]   Philip J. Higgins and Kirill C. H. Mackenzie. "Duality for base-changing morphisms of vector bundles, modules, Lie algebroids and Poisson structures". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 114 (1993). DOI: 10.1017/S0305004100071760.

[JR13]   Michael Johnson and Robert Rosebrugh. "Delta lenses and opfibrations". In: *Electronic Communications of the EASST* 57 (2013). DOI: 10.14279/tuj.eceasst.57.875.

[Kel05]   G.M. Kelly. "Basic Concepts of Enriched Category Theory". In: *Reprints in Theory and Applications of Categories* 10 (2005). URL: http://www.tac.mta.ca/tac/reprints/articles/10/tr10abs.html.

[Law02]   F. W. Lawvere. "Metric spaces, generalized logic and closed categories". In: *Reprints in Theory and Applications of Categories* 1 (2002). URL: http://www.tac.mta.ca/tac/reprints/articles/1/tr1abs.html.

[Par20]   Robert Paré. "Retrocells Redux". MIT Categories Seminar. 2020. URL: https://www.mscs.dal.ca/~pare/RetroCellsMIT.pdf.

[Per21]   Paolo Perrone. *Lifting couplings in Wasserstein spaces*. 2021. arXiv: 2110.06591.

[Spi20]   David I. Spivak. *Poly: An abundant categorical setting for mode-dependent dynamics*. 2020. arXiv: 2005.01894.

[Spi22]   David I. Spivak. *Functorial aggregation*. 2022. arXiv: 2111.10968.

[SN22]   David I. Spivak and Nelson Niu. *Polynomial Functors: A General Theory of Interaction*. 2022. URL: https://topos.site/poly-book.pdf.

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