

# A diagrammatic approach to symmetric lenses

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Lenses are a mathematical structure for maintaining consistency between a pair of systems. In their ongoing research program, Johnson and Rosebrugh have sought to unify the treatment of symmetric lenses with spans of asymmetric lenses. This paper presents a diagrammatic approach to symmetric lenses between categories, through representing the propagation operations with Mealy morphisms. The central result of this paper is to demonstrate that the bicategory of symmetric lenses is locally adjoint to the bicategory of spans of asymmetric lenses, through constructing an explicit adjoint triple between the hom-categories.

## 1 Introduction

Lenses are a mathematical structure which model synchronisation between a pair of systems. Lenses have been actively studied in both computer science and category theory since the seminal paper [7], and now play an important role in a diverse range of applications including bidirectional transformations, model-driven engineering, database view-updating, systems interoperations, data sharing, and functional programming.

While typically lenses are used to describe *asymmetric* relationships between systems, many of these examples are better understood as special cases of *symmetric lenses*. Since the introduction of symmetric lenses in the paper [9], there has been a significant research program lead by Johnson and Rosebrugh (see [12, 13, 14, 15, 16]) to unify their treatment with spans of asymmetric lenses. However, despite this research revealing numerous important aspects of symmetric lenses, many constructions appear ad hoc by relying upon justification from applications, and remain without a robust category-theoretic foundation.

This paper develops a diagrammatic approach to symmetric lenses in category theory, which clarifies and generalises the previous results by Johnson and Rosebrugh. Symmetric lenses are characterised as a pair of *Mealy morphisms*, and may be represented as certain diagrams in  $\text{Cat}$ . The main result demonstrates, for a pair of systems  $A$  and  $B$ , an adjoint triple between the category of symmetric lenses and the category of spans of asymmetric lenses.

$$\text{SymLens}(A, B) \begin{array}{c} \longleftarrow \\ \perp \\ \longleftarrow \\ \perp \\ \longrightarrow \end{array} \text{SpnLens}(A, B) \quad (1.1)$$

Furthermore, these adjunctions characterise  $\text{SymLens}(A, B)$  as both a reflective and coreflective subcategory of  $\text{SpnLens}(A, B)$ , and underlie local adjunctions between the corresponding bicategories  $\text{SymLens}$  and  $\text{SpnLens}$ .

This paper treats a system as a category, whose objects are the *states* of the system, and whose morphisms are the *updates* (or transitions) between states of the system. In the paper [5], *asymmetric*

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*delta lenses* were introduced as the maps between systems, which propagate updates in one system to updates in another. A close category-theoretic study of delta lenses appeared in [11], and in [2] it was discovered that they may be understood in terms of functors and cofunctors. In the ACT2019 paper [3], delta lenses were generalised to internal category theory, and more importantly, it was shown that an asymmetric delta lens may be represented as a certain commutative diagram in  $\mathcal{C}at$ .

The focus of this paper is *symmetric delta lenses*, introduced in [6], and their relationship with spans of asymmetric delta lenses. While the key results are concentrated on the theoretical foundation of these structures, this work also contains important benefits towards applications, from simplifying the use, and further study, of lenses.

## Overview of the paper

Section 2 reviews the different kinds of morphisms between categories which are later used to define lenses. The definitions of discrete opfibration, bijective-on-objects functor, and fully faithful functor are recalled, as well as the less familiar definitions of cofunctor (see [8, 1]) and Mealy morphism (see [18]; also known as a *two-dimensional partial map* in [17, 19]). In Lemma 2.6 and Lemma 2.11, cofunctors and Mealy morphisms are faithfully represented as spans in  $\mathcal{C}at$ . Note that Mealy morphisms in this paper are slightly different from [18, Example 3], as there is no requirement for the functor component to be “objectwise constant on the fibres”. The bicategory  $\mathcal{M}eal$  of small categories and Mealy morphisms is equivalent to the bicategory  $\mathcal{M}nd(\mathcal{S}pan)$  of monads and lax monad morphisms in the bicategory of spans.

In Section 3, the definition of an asymmetric lens is recalled from [5, 11], and their characterisation from [3] as diagrams in  $\mathcal{C}at$  is stated in Lemma 3.2. While the category  $\mathcal{L}ens$  of small categories and lenses does not admit all pullbacks, it is proved in Proposition 3.4 that the category  $\mathcal{L}ens(B)$ , of lenses over a base category  $B$ , has products. Using this proposition, the bicategory  $\mathcal{S}pn\mathcal{L}ens$  of small categories and spans of asymmetric lenses is constructed. From the perspective of applications, the bicategory  $\mathcal{S}pn\mathcal{L}ens$  allows the modelling of updates between systems which cannot synchronise directly, but instead depend on some intermediary system.

Section 4 presents a concise construction of the bicategory  $\mathcal{S}ym\mathcal{L}ens$  of small categories and symmetric lenses, using the bicategory  $\mathcal{M}eal$ . A symmetric lens between a pair of systems may be understood as a set of *correspondences* between the states of the systems, together with a pair of Mealy morphisms which propagate the system updates in each direction. Informally, the “symmetric” aspect of symmetric lenses may be understood in the context of dagger categories, through a canonical family of functors  $\dagger: \mathcal{S}ym\mathcal{L}ens(A, B) \rightarrow \mathcal{S}ym\mathcal{L}ens(B, A)$  which take the opposite of a symmetric lens.

In Section 5, the precise categorical relationship between  $\mathcal{S}pn\mathcal{L}ens$  and  $\mathcal{S}ym\mathcal{L}ens$  is presented by the adjoint triple in Theorem 5.1. The proof relies on the diagrammatic approach to symmetric lenses in an essential way, and reveals several aspects of [15, Theorem 40] which were hidden by an unnecessary equivalence relation.

## 2 Background

Let  $\mathcal{C}at$  denote the category of small categories and functors. There are three classes of functors that will be of particular interest in this paper.

**Definition 2.1.** A functor  $f: A \rightarrow B$  is a *discrete opfibration* if for all objects  $a \in A$  and morphisms  $u: fa \rightarrow b \in B$ , there exists a unique morphism  $\varphi(a, u): a \rightarrow p(a, u)$  in  $A$  such that  $f\varphi(a, u) = u$ . The notation  $p(a, u)$  is used to denote the object  $\text{cod}(\varphi(a, u))$ . Let  $\mathcal{D}$  denote the class of discrete opfibrations.

**Definition 2.2.** A functor is *bijective-on-objects* if its object assignment is a bijection. Let  $\mathcal{E}$  denote the class of bijective-on-objects functors.

**Definition 2.3.** A functor  $f: A \rightarrow B$  is *fully faithful* if for all objects  $a, a' \in A$  and morphisms  $u: fa \rightarrow fa' \in B$ , there exists a unique morphism  $w: a \rightarrow a'$  in  $A$  such that  $fw = u$ . Let  $\mathcal{M}$  denote the class of fully faithful functors.

There is a well-known orthogonal factorisation system  $(\mathcal{E}, \mathcal{M})$  on  $\mathcal{Cat}$ , called the *bo-ff factorisation system*, in which every functor factorises into a bijective-on-objects functor followed by a fully faithful functor. The *image* of a functor  $f: A \rightarrow B$  is a category  $I_f$  whose objects are those of  $A$ , and whose morphisms are triples  $(a, u, a'): a \rightarrow a'$  where  $a, a' \in A$  and  $u: fa \rightarrow fa' \in B$ . The functor  $f: A \rightarrow B$  factorises through the image as follows:

$$\begin{array}{ccccc}
 A & \longrightarrow & I_f & \longrightarrow & B \\
 a & \cdots & a & \cdots & fa \\
 \downarrow w & & \downarrow (a, fw, a') & & \downarrow fw \\
 a' & \cdots & a' & \cdots & fa'
 \end{array} \tag{2.1}$$

The universal property of the bo-ff factorisation system may be stated as follows. Given a commutative square of functors,

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 e \downarrow & \nearrow h & \downarrow m \\
 C & \xrightarrow{g} & D
 \end{array} \tag{2.2}$$

where  $e$  is bijective-on-objects and  $m$  is fully faithful, there exists a unique functor  $h: C \rightarrow B$  such that  $h \circ e = f$  and  $m \circ h = g$ . In particular, note that this universal property defines the image  $I_f$  uniquely up to isomorphism.

**Definition 2.4** (See [1]). Let  $A$  and  $B$  be categories. A *cofunctor*  $\varphi: B \rightarrow A$  consists of an assignment on objects  $\varphi: \text{ob}(B) \rightarrow \text{ob}(A)$  together with an operation assigning each pair  $(a, u)$ , where  $a \in A$  and  $u: \varphi a \rightarrow b \in B$ , to a morphism  $\varphi(a, u): a \rightarrow p(a, u)$  in  $A$ , satisfying the axioms:

- (1)  $\varphi p(a, u) = b$
- (2)  $\varphi(a, 1_{\varphi a}) = 1_a$
- (3)  $\varphi(a, v \circ u) = \varphi(p(a, u), v) \circ \varphi(a, u)$

The notation  $p(a, u)$  is used to denote the object  $\text{cod}(\varphi(a, u))$ .

**Example 2.5.** Every discrete opfibration  $A \rightarrow B$  yields a cofunctor  $B \rightarrow A$ , and every bijective-on-objects functor  $A \rightarrow B$  yields a cofunctor  $A \rightarrow B$ .

Let  $\mathcal{Cof}$  denote the category of small categories and cofunctors. Given cofunctors  $\gamma: C \rightarrow B$  and  $\varphi: B \rightarrow A$ , their composite  $\varphi \circ \gamma: C \rightarrow A$  may be understood from the following diagram:

$$\begin{array}{ccccc}
 C & \xrightarrow{\quad} & B & \xrightarrow{\quad} & A \\
 \gamma\varphi a & \cdots & \varphi a & \cdots & a \\
 u \downarrow & & \downarrow \gamma(\varphi a, u) & & \downarrow \varphi(a, \gamma(\varphi a, u)) \\
 c & \cdots & q(\varphi a, u) & \cdots & p(a, \gamma(\varphi a, u))
 \end{array} \tag{2.3}$$

There is an orthogonal factorisation system  $(\mathcal{D}^{\text{op}}, \mathcal{E})$  on  $\mathcal{C}\text{of}$ , in which every cofunctor factorises into a discrete opfibration (taken in the opposite direction) followed by a bijective-on-objects functor. The *image* of a cofunctor  $\varphi: B \dashrightarrow A$  is a category  $\Lambda$  whose objects are those of  $A$ , and whose morphisms are pairs  $(a, u): a \rightarrow p(a, u)$ , where  $a \in A$  and  $u: \varphi a \rightarrow b \in B$ . The cofunctor  $\varphi: B \dashrightarrow A$  factorises through the image as follows:

$$\begin{array}{ccccccc}
 B & \xrightarrow{\quad} & \Lambda & \xrightarrow{\quad} & A & & \\
 \varphi a & \cdots & a & \cdots & a & & \\
 u \downarrow & & \downarrow (a, u) & & \downarrow \varphi(a, u) & & \\
 b & \cdots & p(a, u) & \cdots & p(a, u) & & 
 \end{array} \tag{2.4}$$

Notice that the cofunctor  $B \dashrightarrow \Lambda$  describes a discrete opfibration  $\Lambda \rightarrow B$ , and the cofunctor  $\Lambda \dashrightarrow A$  describes an identity-on-objects functor  $\Lambda \rightarrow A$ .

**Lemma 2.6.** *Given a cofunctor  $\varphi: B \dashrightarrow A$  there is a span of functors,*

$$\begin{array}{ccc}
 & \Lambda & \\
 \overline{\varphi} \swarrow & & \searrow \varphi \\
 B & & A
 \end{array} \tag{2.5}$$

where  $\overline{\varphi}$  is a discrete opfibration and  $\varphi$  is identity-on-objects.

If  $\mathcal{C}\text{of}$  is understood as a locally-discrete 2-category, Lemma 2.6 provides a way of constructing a locally fully faithful, identity-on-objects pseudofunctor  $\mathcal{C}\text{of} \rightarrow \text{Span}(\text{Cat})$ . From now on cofunctors will always be given by their span representation (2.5).

**Definition 2.7** (See [18]). Let  $A$  and  $B$  be categories. A *Mealy morphism*  $A \dashrightarrow B$  consists of a discrete category  $X_0$  together with a span of functors  $(g_0, X_0, f_0): A \dashrightarrow B$  and operations assigning each pair  $(x, u)$ , where  $x \in X_0$  and  $u: g_0 x \rightarrow a \in A$ , to an object  $q(x, u)$  in  $X_0$  and a morphism  $f(x, u): f_0 x \rightarrow f_0 q(x, u)$  in  $B$ , satisfying the axioms:

- (1)  $g_0 q(x, u) = a$
- (2)  $q(x, 1_{g_0 x}) = x$  and  $f(x, 1_{g_0 x}) = 1_{f_0 x}$
- (3)  $q(x, v \circ u) = q(q(x, u), v)$  and  $f(x, v \circ u) = f(q(x, u), v) \circ f(x, u)$

**Example 2.8.** Every functor  $A \rightarrow B$  yields a Mealy morphism  $A \dashrightarrow B$ , and every cofunctor  $B \rightarrow A$  yields a Mealy morphism  $B \dashrightarrow A$ .

**Example 2.9** (Example 4 in [18]). Given a pair of sets  $A$  and  $B$ , a Mealy morphism between free monoids  $A^*$  and  $B^*$  is exactly a *Mealy machine* with input alphabet  $A$  and output alphabet  $B$ .

Let  $\mathcal{M}\text{eal}$  denote the *bicategory* of small categories and Mealy morphisms. Unlike the special cases functors and cofunctors, composition of Mealy morphisms is not strictly associative, since the structure involves a span of functions. There are two possible notions of 2-cell between Mealy morphisms; this paper uses the stricter notion as given below.

**Definition 2.10.** Let  $(X_0, g_0, f_0, q, f)$  and  $(Y_0, k_0, h_0, p, h)$  be Mealy morphisms  $A \dashrightarrow B$ . A *map of Mealy morphisms* consists of a morphism of spans,

$$\begin{array}{ccccc}
 & X_0 & & & \\
 g_0 \swarrow & \downarrow m & \searrow f_0 & & \\
 A & & B & & \\
 k_0 \swarrow & \downarrow p & \searrow h_0 & & \\
 & Y_0 & & & 
 \end{array} \tag{2.6}$$

such that  $mq(x, u) = p(mx, u)$  and  $f(x, u) = h(mx, u)$  for each pair  $(x, u)$ , where  $x \in X_0$  and  $u: g_0x \rightarrow a \in A$ .

Analogous to the orthogonal factorisation system on  $\mathcal{Cof}$ , every Mealy morphism factorises into a discrete opfibration followed by a functor. Using the notation of Definition 2.10, the *image* of a Mealy morphism  $A \rightrightarrows B$  is a category  $X$ , whose set of objects is  $X_0$  and whose morphisms are pairs  $(x, u): x \rightarrow q(x, u)$ , where  $x \in X_0$  and  $u: g_0x \rightarrow a$ . The factorisation of a Mealy morphism may then be described by the following diagram:

$$\begin{array}{ccccccc}
 A & \xrightarrow{\quad / \quad} & X & \xrightarrow{\quad / \quad} & B & & \\
 g_0x & \cdots & x & \cdots & f_0x & & \\
 \downarrow u & & \downarrow (x,u) & & \downarrow f(x,u) & & \\
 a & \cdots & q(x,u) & \cdots & f_0q(x,u) & & 
 \end{array} \tag{2.7}$$

Notice that the Mealy morphism  $A \rightrightarrows X$  describes a discrete opfibration  $X \rightarrow A$ , and the Mealy morphism  $X \rightrightarrows B$  describes a functor  $X \rightarrow B$ .

**Lemma 2.11.** *Given a Mealy morphism  $A \rightrightarrows B$  there is a span of functors,*

$$\begin{array}{ccc}
 & X & \\
 \bar{g} \swarrow & & \searrow f \\
 A & & B
 \end{array} \tag{2.8}$$

where  $\bar{g}$  is a discrete opfibration.

Lemma 2.11 provides a way of constructing a locally fully faithful, identity-on-objects pseudofunctor  $\mathcal{Meal} \rightarrow \text{Span}(\text{Cat})$ . From now on Mealy morphisms will always be understood by their span representation (2.8). It is also worth noting that every Mealy morphism may also be factorised into a cofunctor followed by a fully faithful functor. These two possible factorisations would amount to a kind of ternary factorisation system  $(\mathcal{D}^{\text{op}}, \mathcal{E}, \mathcal{M})$  on  $\mathcal{Meal}$ , however this observation won't be pursued in this paper.

### 3 Spans of asymmetric lenses

The goal of this section is to introduce the following three structures:

- ◇ The category  $\mathcal{Lens}$  of small categories and (asymmetric) lenses;
- ◇ The category  $\mathcal{Lens}(B)$  of lenses over a base category  $B$ ;
- ◇ The bicategory  $\text{Spn}\mathcal{Lens}$  of small categories and spans of lenses.

It is well-known that  $\mathcal{Lens}$  does not have all pullbacks, which complicates the usual construction of the bicategory of spans. The main obstruction is that while every cospan in  $\mathcal{Lens}$  admits a canonical cone, the universal property of the pullback does not hold. However, for any small category  $B$ , there is a suitably defined category  $\mathcal{Lens}(B)$  which does admit cartesian products. Together these categories allow for the construction of a suitable bicategory  $\text{Spn}\mathcal{Lens}$ , whose morphisms are spans in  $\mathcal{Lens}$  but whose 2-cells are defined by morphisms in  $\mathcal{Lens}(B)$ .

**Definition 3.1.** An (asymmetric) lens  $(f, \varphi): A \rightrightarrows B$  consists of a functor  $f: A \rightarrow B$  together with a function,

$$(a \in A, u: fa \rightarrow b) \quad \longmapsto \quad \varphi(a, u): a \rightarrow p(a, u)$$

satisfying the axioms:

- (1)  $f\varphi(a, u) = u$
- (2)  $\varphi(a, 1_{fa}) = 1_a$
- (3)  $\varphi(a, v \circ u) = \varphi(p(a, u), v) \circ \varphi(a, u)$

Equivalently, an asymmetric lens consists of a functor  $f: A \rightarrow B$  together with a cofunctor  $\varphi: B \rightarrow A$  such that  $fa = \varphi a$  and  $f\varphi(a, u) = u$ .

The functor and cofunctor components of asymmetric lens are usually known as the GET and the PUT, respectively. The three axioms above also correspond to the PUTGET, GETPUT, and PUTPUT laws, respectively.

**Lemma 3.2.** *Given a lens  $(f, \varphi): A \rightleftarrows B$  there is a commutative diagram of functors,*

$$\begin{array}{ccc} & \Lambda & \\ \varphi \swarrow & & \searrow \bar{\varphi} \\ A & \xrightarrow{f} & B \end{array} \quad (3.1)$$

where  $\varphi$  is an identity-on-objects functor and  $\bar{\varphi}$  is a discrete opfibration.

Like cofunctors and Mealy morphisms, a lens will always be understood by its diagrammatic representation (3.1) in  $\mathcal{Cat}$ . Let  $\mathcal{Lens}$  be the category of small categories and lenses. Composition of lenses is given by composing the respective functor and cofunctor components, and the representation (3.1) of the composite may be understood by the following diagram:

$$\begin{array}{ccccc} & & \Lambda \times_B \Omega & & \\ & & \swarrow & \searrow & \\ & \Lambda & & & \Omega \\ \varphi \swarrow & & \searrow \bar{\varphi} & \gamma \swarrow & \searrow \bar{\gamma} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array} \quad (3.2)$$

Given a pair of lenses  $(f, \varphi): A \rightleftarrows B$  and  $(g, \gamma): C \rightleftarrows B$  forming a cospan in  $\mathcal{Lens}$ ,

$$\begin{array}{ccccc} & \Lambda & & \Omega & \\ \varphi \swarrow & & \searrow \bar{\varphi} & \bar{\gamma} \swarrow & \searrow \gamma \\ A & \xrightarrow{f} & B & \xleftarrow{g} & C \end{array} \quad (3.3)$$

there is a canonical cone, or “fake pullback”, given by the span in  $\mathcal{Lens}$ :

$$\begin{array}{ccccc} & A \times_B \Omega & & \Lambda \times_B C & \\ \pi_0 \swarrow & & \searrow 1 \times \gamma & \varphi \times 1 \swarrow & \searrow \pi_1 \\ A & \xleftarrow{\pi_0} & A \times_B C & \xrightarrow{\pi_1} & C \end{array} \quad (3.4)$$

Note that this “fake pullback” diagram is sent to a genuine pullback via the forgetful functor  $\mathcal{Lens} \rightarrow \mathcal{Cat}$ . The category  $\mathcal{Lens}$  also has the same terminal object as  $\mathcal{Cat}$ , and fake pullbacks over the terminal yields a semi-cartesian monoidal structure on  $\mathcal{Lens}$ .

The reason (3.4) fails, in general, to be a genuine pullback in  $\mathcal{Lens}$  is that the corresponding universal property is not satisfied. However, recall that pullbacks in  $\mathcal{Cat}$  are the same as products in a slice category  $\mathcal{Cat}/B$  for some small category  $B$ . While the slice category  $\mathcal{Lens}/B$  is not useful, there is a suitable category  $\mathcal{Lens}(B)$  with cartesian products, together with a product-preserving functor  $\mathcal{Lens}(B) \rightarrow \mathcal{Cat}/B$ , that provides the “fake pullbacks” in  $\mathcal{Lens}$  with a universal property.

**Definition 3.3.** The category  $\mathcal{Lens}(B)$  of *lenses over a base category*  $B$  has objects given by lenses with codomain  $B$ , and morphisms  $(f, \varphi) \rightarrow (g, \gamma)$  given by commutative diagrams of the form:

$$\begin{array}{ccc}
 \Lambda & \xrightarrow{\bar{h}} & \Omega \\
 \varphi \downarrow & \searrow \bar{\varphi} & \swarrow \bar{\gamma} \downarrow \gamma \\
 A & \xrightarrow{h} & C \\
 & \searrow f & \swarrow g \\
 & & B
 \end{array} \tag{3.5}$$

Note that only the functor  $h: A \rightarrow C$  above need be specified; the functor  $\bar{h}: \Lambda \rightarrow \Omega$  will always be uniquely induced from  $h$ , however may not (in general) make the back square in (3.5) commute. The above definition of  $\mathcal{Lens}(B)$  is motivated as a generalisation of the category of  $\mathcal{SOpf}(B)$  of split opfibrations and cleavage-preserving functors, which is obtained as a full subcategory. An variant of  $\mathcal{Lens}(B)$  has also been considered in [11] as the category of algebras for a semi-monad on  $\mathcal{Cat}/B$ .

**Proposition 3.4.** *The category  $\mathcal{Lens}(B)$  has products, for all small categories  $B$ .*

*Proof.* Consider a pair of lenses in  $\mathcal{Lens}(B)$  as depicted in (3.3). Their product is given by the lens,

$$\begin{array}{ccc}
 \Lambda \times_B \Omega & & \\
 \varphi \times \gamma \swarrow & & \searrow \bar{\varphi} \pi_0 = \bar{\gamma} \pi_1 \\
 A \times_B C & \xrightarrow{f \pi_0 = g \pi_1} & B
 \end{array} \tag{3.6}$$

which is equal the composite of the appropriate lenses in (3.3) and (3.4). The product projections are given by the following diagrams:

$$\begin{array}{ccc}
 \Lambda & \xleftarrow{\pi_0} & \Lambda \times_B \Omega \\
 \varphi \downarrow & \searrow \bar{\varphi} & \swarrow \bar{\varphi} \pi_0 \downarrow \varphi \times \gamma \\
 A & \xleftarrow{\pi_0} & A \times_B C \\
 & \searrow f & \swarrow f \pi_0 \\
 & & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Lambda \times_B \Omega & \xrightarrow{\pi_1} & \Omega \\
 \varphi \times \gamma \downarrow & \searrow \bar{\gamma} \pi_1 & \swarrow \bar{\gamma} \downarrow \gamma \\
 A \times_B C & \xrightarrow{\pi_1} & C \\
 & \searrow g \pi_1 & \swarrow g \\
 & & B
 \end{array} \tag{3.7}$$

It is not difficult to show that the lens (3.6) also satisfies the universal property of the product in the category  $\mathcal{Lens}(B)$ . □

Proposition 3.4 shows that the fake pullbacks constructed in  $\mathcal{Lens}$  actually have a universal property with respect to the morphisms in  $\mathcal{Lens}(B)$ , for the appropriate small category  $B$ . Now consider the family of functors  $\mathcal{Lens}(B) \rightarrow \mathcal{Cat}$  which assign each lens to its domain category, and each morphism (3.5) to the corresponding functor between domains.

**Definition 3.5.** Let  $\mathcal{SpnLens}$  be the bicategory of *spans of asymmetric lenses*, whose objects are small categories, and whose hom-categories  $\mathcal{SpnLens}(A, B)$  are constructed by the pullback:

$$\begin{array}{ccc}
 & \mathcal{SpnLens}(A, B) & \\
 \swarrow & \downarrow & \searrow \\
 \mathcal{Lens}(A) & & \mathcal{Lens}(B) \\
 \searrow & & \swarrow \\
 & \mathcal{Cat} &
 \end{array} \tag{3.8}$$

An object in  $\text{SpnLens}(A, B)$  is a span of asymmetric lenses from  $A$  to  $B$ , and a morphism is given by a functor  $X \rightarrow X'$ , together with induced functors  $\Omega \rightarrow \Omega'$  and  $\Lambda \rightarrow \Lambda'$ , such that each face (including the two outer squares) in the following diagram commute:

$$\begin{array}{ccccc}
 \Omega & \xrightarrow{\quad} & X & \xleftarrow{\quad} & \Lambda \\
 \downarrow & \searrow & \downarrow & \swarrow & \downarrow \\
 & & A & & B \\
 \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
 \Omega' & \xrightarrow{\quad} & X' & \xleftarrow{\quad} & \Lambda'
 \end{array} \tag{3.9}$$

Horizontal composition is given by fake pullback of lenses, followed by lens composition of the projections with the appropriate legs of the span. Horizontal composition is associative up to natural isomorphism with respect to the morphisms (3.9) above.

There is an identity-on-objects pseudofunctor  $\mathcal{Lens} \rightarrow \text{SpnLens}$  which takes a lens  $A \rightleftharpoons B$  to the right leg of a span of lenses from  $A$  to  $B$ , with left leg given by the identity lens.

The construction of the bicategory  $\text{SpnLens}$  is a generalisation of a *category* previously defined in [13, 15]. This category has objects given by small categories, and certain *equivalence classes* of spans of asymmetric lenses as morphisms. Removing the equivalence relation and considering the appropriate 2-cells naturally gives rise to the bicategory  $\text{SpnLens}$  considered here.

## 4 Symmetric lenses

The goal of this section is to introduce the bicategory  $\text{SymLens}$  of small categories and symmetric lenses.

Consider the family of functors  $U_{A,B}: \text{Meal}(A, B) \rightarrow \text{Span}(\text{Cat})(A, B)$  with the assignment on Mealy morphisms (Definition 2.10) stated for the span representation (2.8) as follows:

$$\begin{array}{ccc}
 \begin{array}{ccc} & X & \\ \bar{g} \swarrow & & \searrow f \\ A & & B \end{array} & \longmapsto & \begin{array}{ccc} & X_0 & \\ g_0 \swarrow & & \searrow f_0 \\ A & & B \end{array}
 \end{array} \tag{4.1}$$

This functor is given by pre-composing the legs of the span with the canonical identity-on-objects functor from the discrete category  $X_0 \rightarrow X$ . Furthermore, consider the family of functors  $\dagger_{A,B}: \text{Span}(\text{Cat})(B, A) \rightarrow \text{Span}(\text{Cat})(A, B)$  which send each span  $(f, X, g): B \rightarrow A$  to its reverse span  $(g, X, f): A \rightarrow B$ .

**Definition 4.1.** Let  $\text{SymLens}$  be the bicategory of *symmetric lenses*, whose objects are small categories, and whose hom-categories  $\text{SymLens}(A, B)$  are constructed by the pullback:

$$\begin{array}{ccc}
 & \text{SymLens}(A, B) & \\
 \swarrow & \downarrow \sphericalangle & \searrow \\
 \text{Meal}(A, B) & & \text{Meal}(B, A) \\
 \swarrow U_{A,B} & & \nwarrow \dagger_{A,B} \circ U_{B,A} \\
 & \text{Span}(\text{Cat})(A, B) &
 \end{array} \tag{4.2}$$



An object in  $\text{Sym}\mathcal{L}ens(A, B)$  is a symmetric lens, and may be depicted by a pair of spans:

$$\begin{array}{ccc}
 & X^+ & \\
 \bar{g} \swarrow & & \searrow f \\
 A & & B \\
 \nwarrow g & & \nearrow \bar{f} \\
 & X^- &
 \end{array}
 \tag{4.3}$$

The 2-cells are given by the corresponding maps of Mealy morphisms, and horizontal composition is also inherited from  $\mathcal{M}eal$ .

*Notation 4.2.* In the diagram (4.3), the upper span is a Mealy morphism  $A \rightrightarrows B$ , while the lower span is a Mealy morphism  $B \rightrightarrows A$ . As the notation suggests, both  $X^+$  and  $X^-$  are categories with the same discrete category of objects  $X_0$ . Moreover, the following diagrams commute:

$$\begin{array}{ccc}
 & X_0 & \\
 \swarrow & & \searrow \langle g_0, f_0 \rangle \\
 X^+ & \xrightarrow{\langle \bar{g}, f \rangle} & A \times B
 \end{array}
 \qquad
 \begin{array}{ccc}
 & X_0 & \\
 \swarrow & & \searrow \langle g_0, f_0 \rangle \\
 X^- & \xrightarrow{\langle g, \bar{f} \rangle} & A \times B
 \end{array}
 \tag{4.4}$$

Taking these diagrams together with (4.3), a symmetric lens may be completely described by the following commutative diagram of functors,

$$\begin{array}{ccc}
 & X^+ & \\
 \bar{g} \swarrow & \uparrow & \searrow f \\
 A & X_0 & B \\
 \nwarrow g & \downarrow & \nearrow \bar{f} \\
 & X^- &
 \end{array}
 \tag{4.5}$$

where  $\bar{g}$  and  $\bar{f}$  are discrete opfibrations. However, for the remainder of the paper a symmetric lens will be depicted by a diagram of the form (4.3) for simplicity.

There is an identity-on-objects pseudofunctor  $\mathcal{L}ens \rightarrow \text{Sym}\mathcal{L}ens$  with the following assignment on morphisms:

$$\begin{array}{ccc}
 & \Lambda & \\
 \varphi \swarrow & & \searrow \bar{\varphi} \\
 A & \xrightarrow{f} & B
 \end{array}
 \quad \mapsto \quad
 \begin{array}{ccc}
 & A & \\
 1 \swarrow & & \searrow f \\
 A & & B \\
 \nwarrow \varphi & & \nearrow \bar{\varphi} \\
 & \Lambda &
 \end{array}
 \tag{4.6}$$

Note that the discrete opfibration  $\bar{\varphi}$  above would usually be denoted by  $\bar{f}$  with the notational convention for symmetric lenses. From this pseudofunctor, symmetric lenses may be seen as a generalisation of asymmetric lenses. In  $\text{Sym}\mathcal{L}ens$  morphisms are pairs of suitable Mealy morphisms, while in  $\mathcal{L}ens$  this must be a functor/cofunctor pair. However there is also a loss of information in (4.6), as a symmetric lens no longer encodes the commutativity condition of the corresponding asymmetric lens.

The construction of the bicategory  $\text{Sym}\mathcal{L}ens$  is a generalisation of a *category* previously defined in [13, 15]. This category has objects given by small categories, and certain *equivalence classes* of symmetric lenses (called *fb-lenses*) as morphisms. Removing the equivalence relation and considering the appropriate 2-cells yields the bicategory  $\text{Spn}\mathcal{L}ens$  considered here.

## 5 An adjoint triple

This section presents the main theorem of the paper.

**Theorem 5.1.** *Let  $A$  and  $B$  be small categories. Then there exists adjoint triple  $L \dashv M \dashv R$  between the category of symmetric lenses and the category of spans of asymmetric lenses,*

$$\text{Sym}\mathcal{L}\text{ens}(A, B) \begin{array}{c} \xleftarrow{L} \\ \xleftarrow{\perp} \\ \xleftarrow{\perp} \\ \xrightarrow{R} \end{array} \text{Spn}\mathcal{L}\text{ens}(A, B) \quad (5.1)$$

such that  $R$  is reflective and  $L$  is coreflective (that is,  $ML = MR = 1$ ).

The functor  $M: \text{Spn}\mathcal{L}\text{ens}(A, B) \rightarrow \text{Sym}\mathcal{L}\text{ens}(A, B)$  is defined on objects as follows:

$$\begin{array}{ccc} \Omega & \xrightarrow{\gamma} & X & \xleftarrow{\varphi} & \Lambda \\ & \searrow \bar{\gamma} & \swarrow s & \searrow f & \swarrow \bar{\varphi} \\ & & A & & B \end{array} \quad \mapsto \quad \begin{array}{ccc} & \Omega & \\ \bar{\gamma} \swarrow & & \searrow f\gamma \\ A & & B \\ & \nwarrow s\varphi & \nearrow \bar{\varphi} \\ & \Lambda & \end{array} \quad (5.2)$$

Recall that  $\gamma$  and  $\varphi$  are identity-on-objects functors, so  $\Omega$  and  $\Lambda$  have the same objects, and the resulting symmetric lens is well-defined.

To construct the right adjoint  $R$ , consider a symmetric lens given by (4.3). Applying the bo-ff factorisation (2.1) to the functor  $\langle g_0, f_0 \rangle: X_0 \rightarrow A \times B$  yields a diagram:

$$\begin{array}{ccc} & \tilde{X} & \\ e \nearrow & & \searrow m \\ X_0 & \xrightarrow{\langle g_0, f_0 \rangle} & A \times B \end{array} \quad (5.3)$$

This factorisation is chosen such that image  $\tilde{X}$  has the same objects as the discrete category  $X_0$ . Using the universal property (2.2) of the bo-ff factorisation, together with the commutative diagrams (4.4), there exists unique, identity-on-objects functors:

$$\begin{array}{ccc} X_0 & \xrightarrow{e} & \tilde{X} \\ \downarrow & \nearrow \sigma & \downarrow m \\ X^+ & \xrightarrow{\langle \bar{g}, f \rangle} & A \times B \end{array} \quad \begin{array}{ccc} X_0 & \xrightarrow{e} & \tilde{X} \\ \downarrow & \nearrow \tau & \downarrow m \\ X^- & \xrightarrow{\langle g, \bar{f} \rangle} & A \times B \end{array} \quad (5.4)$$

The functor  $R: \text{Sym}\mathcal{L}\text{ens}(A, B) \rightarrow \text{Spn}\mathcal{L}\text{ens}(A, B)$  is defined on objects as follows:

$$\begin{array}{ccc} & X^+ & \\ \bar{g} \swarrow & & \searrow f \\ A & & B \\ & \nwarrow s & \nearrow \bar{f} \\ & X^- & \end{array} \quad \mapsto \quad \begin{array}{ccc} X^+ & \xrightarrow{\sigma} & \tilde{X} & \xleftarrow{\tau} & X^- \\ & \nwarrow \bar{g} & \swarrow \pi_0 m & \swarrow \pi_1 m & \nwarrow \bar{f} \\ & A & & & B \end{array} \quad (5.5)$$

One may notice immediately that the composite  $MR: \text{SymLens}(A, B) \rightarrow \text{SymLens}(A, B)$  is equal to the identity functor. The unit for the adjunction  $M \dashv R$  is constructed using the universal property of the bo-ff factorisation, and is given as follows:

$$\begin{array}{ccccc}
 \Omega & \xrightarrow{\gamma} & X & \xleftarrow{\varphi} & \Lambda \\
 \searrow \bar{\gamma} & & \swarrow g & & \swarrow \bar{\varphi} \\
 & & A & \xleftarrow{\pi_{0m}} & B \\
 \swarrow \bar{\gamma} & & \nwarrow \pi_{1m} & & \swarrow \bar{\varphi} \\
 \Omega & \xrightarrow{\sigma} & \tilde{X} & \xleftarrow{\tau} & \Lambda \\
 \downarrow 1_\Omega & & \downarrow & & \downarrow 1_\Lambda
 \end{array} \tag{5.6}$$

To construct the left adjoint  $L$ , again consider a symmetric lens given by (4.3). Since  $X^+$  and  $X^-$  have the same discrete category of objects  $X_0$ , there is pushout along the identity-on-objects functors given by:

$$\begin{array}{ccc}
 X_0 & \longrightarrow & X^- \\
 \downarrow & \lrcorner & \downarrow \iota_1 \\
 X^+ & \xrightarrow{\iota_0} & X^+ \sqcup_{X_0} X^-
 \end{array} \tag{5.7}$$

For brevity, let  $\hat{X} := X^+ \sqcup_{X_0} X^-$ . Note that identity-on-objects functors are stable under pushout, so both  $\iota_0$  and  $\iota_1$  are also identity-on-objects functors. The functor  $R: \text{SymLens}(A, B) \rightarrow \text{SpnLens}(A, B)$  is defined on objects as follows:

$$\begin{array}{ccc}
 \begin{array}{ccc} X^+ & & \\ \bar{g} \swarrow & & \searrow f \\ A & & B \\ \swarrow g & & \searrow \bar{f} \\ X^- & & \end{array} & \longmapsto & \begin{array}{ccccc} X^+ & \xrightarrow{\iota_0} & \hat{X} & \xleftarrow{\iota_1} & X^- \\ \bar{g} \swarrow & & \swarrow [\bar{g}, g] & & \swarrow [f, \bar{f}] \\ A & & & & B \end{array}
 \end{array} \tag{5.8}$$

One may notice immediately that the composite  $ML: \text{SymLens}(A, B) \rightarrow \text{SymLens}(A, B)$  is equal to the identity functor. The counit for the adjunction  $L \dashv M$  is constructed using the universal property of the pushout, and is given as follows:

$$\begin{array}{ccccc}
 \Omega & \xrightarrow{\iota_0} & \hat{X} & \xleftarrow{\iota_1} & \Lambda \\
 \searrow \bar{\gamma} & & \swarrow [\bar{\gamma}, g\varphi] & & \swarrow [f\gamma, \bar{\varphi}] \\
 & & A & \xleftarrow{g} & B \\
 \swarrow \bar{\gamma} & & \nwarrow [\gamma, \varphi] & & \swarrow \bar{\varphi} \\
 \Omega & \xrightarrow{\gamma} & X & \xleftarrow{\varphi} & \Lambda \\
 \downarrow 1_\Omega & & \downarrow & & \downarrow 1_\Lambda
 \end{array} \tag{5.9}$$

**Corollary 5.2.** *There exist identity-on-objects pseudofunctors between the bicategory of symmetric lenses and the bicategory of spans of asymmetric lenses,*

$$\begin{array}{ccc}
 & \xrightarrow{L} & \\
 \text{SymLens} & \xleftarrow{M} & \text{SpnLens} \\
 & \xrightarrow{R} &
 \end{array} \tag{5.10}$$

such that  $L$  and  $R$  are locally fully faithful and are locally adjoint to  $M$ .

## 6 Concluding remarks and future work

This paper has established a new category-theoretic foundation for symmetric delta lenses. In contrast to the algebraic approach of Johnson and Rosebrugh, this paper develops a natural diagrammatic approach to symmetric lenses and spans of asymmetric lenses, by using the properties of certain classes of functors. This framework yields significantly simpler definitions (for example, compare the characterisation of a symmetric lens via Mealy morphisms in (4.3) to [15, Definition 7]), and allows for a clearer understanding of the composition of symmetric lenses, which is important for their application in fields such as database view-updating and model-driven engineering.

While symmetric lenses and spans of asymmetric lenses were previously understood in [15] as morphisms in an *isomorphic* pair of categories, the bicategories  $\text{SymLens}$  and  $\text{SpnLens}$  constructed in this paper share a more interesting relationship. The main theorem shows that  $\text{SymLens}(A, B)$  is both a reflective and coreflective subcategory of  $\text{SpnLens}(A, B)$ , which suggests that symmetric lenses are less expressive than spans of asymmetric lenses when modelling update propagation between systems. The subcategory inclusions also provide a way of characterising which spans of asymmetric lenses arise from symmetric lenses: either the functor components of the span are jointly fully faithful (via the right adjoint) or the identity-on-objects functors in the cofunctor components are pushout injections (via the left adjoint). A detailed study of the mathematical implications of the local adjunction between  $\text{SymLens}$  and  $\text{SpnLens}$  is left for future research.

The notion of universal symmetric lenses, as considered in [16, 10], will also be the focus of future work. In the paper [4], explicit conditions for universal asymmetric lenses were established, and it is hoped that these results may be extended to the symmetric setting.

Although this paper has established explicit technical advances towards the understanding of symmetric delta lenses, it also suggests broader goals for the understanding of lenses in applied category theory. Analogous to the transition from functions to relations, this paper further develops the transition from asymmetric lenses to the general setting of symmetric lenses, as pioneered by Johnson and Rosebrugh. Realising this framework with other kinds of lenses in the literature has the potential to capture a wider range of applications and deliver mathematically interesting results.

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