The right-connected completion of a double category*

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A double category is an internal category in the 2-category $\mathcal{C}\mathrm{AT}$, and is called *right-connected* if its identity-assigning map is right adjoint to its codomain-assigning map. Intuitively, a right-connected double category is one in which each vertical morphism has an *underlying* horizontal morphism. An important example of a right-connected double category arises from an *algebraic weak factorisation system* (L, R) on a category $\mathcal{C}$, where the vertical morphisms are the R-algebras and the horizontal morphisms come from $\mathcal{C}$. Riehl [5] showed that this example extends to a 2-functor from the 2-category $\mathcal{A}\mathrm{WFS}_{\mathrm{lax}}$ of algebraic weak factorisation systems to the 2-category $\mathcal{D}\mathrm{BL}$ of double categories, and its essential image was characterised by Bourke and Garner [2] to consist of those right-connected double categories which satisfy a certain monadicity condition. A natural question arises: can we construct an algebraic weak factorisation system from an arbitrary double category?

In this talk, I will introduce the *right-connected completion* $\Gamma(\mathbb{D})$ of a double category $\mathbb{D}$, and provide several instances where $\Gamma(\mathbb{D})$ satisfies the required monadicity condition to induce an algebraic weak factorisation system. In addition, I will exhibit many examples of the right-connected completion of well-known double categories, demonstrating why this completion is also interesting in its own right.

Two different approaches to constructing the right-connected completion will be established. The first approach involves characterising the internal nerve of a right-connected double category via a certain *relative left* 2*-adjoint*, which sends a category $\mathcal{C}$ to the free right-connected double category $\mathbb{R}\mathrm{c}(\mathcal{C})$ relative to the vertical double category $\mathbb{V}(\mathcal{C})$. This allows for an explicit description of $\Gamma(\mathbb{D})$ in terms of its nerve given by $\mathcal{D}\mathrm{BL}(\mathbb{R}\mathrm{c}(-), \mathbb{D}) \colon \Delta^{\mathrm{op}} \to \mathcal{C}\mathrm{AT}$. The second approach involves using comma objects in the slice 2-category $\mathcal{C}\mathrm{AT}/\mathcal{C}$ to construct the cofree left-adjoint-left-inverse on a split epimorphism in $\mathcal{C}\mathrm{AT}$, and applying this to the codomain-assigning map of a double category.

The right-connected completion characterises the 2-category $\mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL}$ of right-connected double categories as a *coreflective* sub-2-category of $\mathcal{D}\mathrm{BL}$, and the counit components $\Gamma(\mathbb{D}) \to \mathbb{D}$ are shown to satisfy a certain comonadicity condition under a mild assumption on $\mathbb{D}$. In this situation, we are able to view vertical morphisms in $\Gamma(\mathbb{D})$ as vertical morphisms in $\mathbb{D}$ equipped with additional *coalgebraic* structure. This highlights an interesting duality with algebraic weak factorisation systems, where the unit components of a *reflective* 2-adjunction between $\mathcal{C}\mathrm{AT}$ and $\mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL}$ satisfy an analogous monadicity condition, and thus allow the vertical morphisms in a right-connected double category to be seen as horizontal morphisms equipped with additional *algebraic* structure.

One of the main applications of the right-connected completion is to present a unified double-categorical framework for the study of *delta lenses* from computer science [4]. The double category of delta lenses [3] arises as the right-connected completion of the double category of categories, functors, and *cofunctors* [1]. This double category satisfies the monadicity condition characterising delta lenses as the R-algebras for an algebraic weak factorisation system on $\mathcal{C}\mathrm{AT}$, as well as the comonadicity condition characterising delta lenses as coalgebras for a comonad on a category of cofunctors.

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