Three approaches to lenses over a base*

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Lenses are functors equipped with a suitable choice of lifts, and split opfibrations are lenses with the property that the chosen lifts are opcartesian. Since lenses were introduced in computer science [2], there has been an ongoing programme to understand them using category theory. Given that lenses directly generalise split opfibrations, it is interesting and fruitful to explore the connections between them, and discover new ways in which the theory of one informs the theory of the other.

In this talk, I will introduce three approaches to the category $\mathsf{Lens}(\mathcal{B})$ of lenses over a base category $\mathcal{B}$, and discuss the relationships between these approaches. I will also examine how the category $\mathsf{SOpf}(\mathcal{B})$ of split opfibrations over a base arises as a full subcategory of $\mathsf{Lens}(\mathcal{B})$ from these different perspectives.

First, I will introduce a *fibred approach* to lenses. The classical Grothendieck construction provides an equivalence between strict functors from $\mathcal{B}$ into $\mathcal{C}at$, and split opfibrations into $\mathcal{B}$. Is there an analogous construction for lenses? One way in which the Grothendieck construction generalises is to an equivalence between ordinary functors into $\mathcal{B}$, and lax double functors from $\mathcal{B}$ into the double category of sets, functions, and spans. A split multi-valued function is a span whose left leg has a chosen section. I will show that the category $\mathsf{Lens}(\mathcal{B})$ is equivalent to the category of lax double functors from $\mathcal{B}$ into the double category of sets, functions, and split multi-valued functions. We will also see how this provides a new viewpoint for understanding the Grothendieck construction.

Second, I will introduce an *algebraic approach* to lenses. Street [4] proved that split opfibrations over $\mathcal{B}$ are equivalent to algebras for a monad on the slice category $\mathcal{C}at/\mathcal{B}$. Based upon this result, Johnson and Rosebrugh [3] demonstrated that lenses arise as certain algebras for a semi-monad on $\mathcal{C}at/\mathcal{B}$, however it remained open as to whether their result could be improved. I will show that the forgetful functor from $\mathsf{Lens}(\mathcal{B})$ to $\mathcal{C}at/\mathcal{B}$, which assigns each lens to its underlying functor, is monadic, establishing that lenses are equivalent to algebras for a monad. I will also discuss the close connection to the previous result of Johnson and Rosebrugh.

Finally, I will introduce a *coalgebraic approach* to lenses. Cofunctors capture the notion of a "suitable choice of lifts" between categories, and may be understood as spans of functors whose left leg is bijective-on-objects and whose right leg is a discrete opfibration. In 2017, Ahman and Uustalu were the first to characterise lenses in terms of functors and cofunctors, and in earlier work [1] they establish a certain method of constructing a lens from a cofunctor. I will show that the forgetful functor from $\mathsf{Lens}(\mathcal{B})$ to $\mathsf{Cof}(\mathcal{B})$, which assigns each lens to its underlying cofunctor, is comonadic, establishing that lenses are equivalent to coalgebras for a comonad. We will see that the right adjoint is exactly the construction previously described by Ahman and Uustalu.

Altogether, these three different approaches to lenses not only clarify past results in the literature, but also provide important new tools and perspectives for current and future work. For instance, the fibred approach yields a concrete way of constructing examples of lenses, which is useful for applications in computer science where this can be an otherwise difficult or ad hoc process. The algebraic approach provides a natural way of understanding pullbacks of lenses, which is important in the theory of symmetric lenses, while the coalgebraic approach reveals how the theory of lenses and split opfibrations may be further generalised internal to any category with certain additional structure.

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References

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