ENRICHED LENSES (EXTENDED ABSTRACT)

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A bidirectional transformation is a specification of when the joint state of two systems should be regarded as consistent, together with a protocol for updating each system to restore consistency in response to a change in the other. Since Pierce et al. introduced the term *lens* for their model of bidirectional transformations for tree-structured data [5], the term has been adopted more widely to describe a menagerie of mathematical structures in applied category theory. In our talk, we will introduce the new notion of *enriched lenses* that subsumes the existing notions of *delta lenses* [4] and *weighted lenses* [8], and give examples of several bases of enrichment for which enriched cofunctors and enriched lenses have an applied interpretation. We will also provide conditions for an enriched cofunctor to be represented by a certain span of enriched functors.

Enriched lenses are a kind of morphism between *enriched categories*. Generalising from the characterisation of a delta lens as a functor and a cofunctor that are compatible with each other [1], an enriched lens is defined as a compatible enriched functor and *enriched cofunctor*.

Definition 1. Let $\mathcal{A}$ and $\mathcal{B}$ be categories enriched in a distributive monoidal category $(\mathcal{V}, \otimes, I)$. An *enriched cofunctor* $(F, \varphi)\colon \mathcal{A} \nrightarrow \mathcal{B}$ consists of a function $F\colon \operatorname{obj}(\mathcal{A}) \to \operatorname{obj}(\mathcal{B})$ and a family

$$\varphi_{a,b}\colon \mathcal{B}(Fa, b) \longrightarrow \sum_{x\in X} \mathcal{A}(a,x) \qquad\qquad X = F^{-1}\{b\}$$

of morphisms in $\mathcal{V}$, such that the following diagrams, which encode preservation of identities and composites, commute.

$$\begin{array}{ccc} I & \xrightarrow{\quad \eta_{Fa} \quad} & \mathcal{B}(Fa, Fa) \\ {\scriptstyle \eta_a}\downarrow & & \downarrow{\scriptstyle \varphi_{a,Fa}} \\ \mathcal{A}(a,a) & \xrightarrow[\quad \iota_a \quad]{} & \displaystyle\sum_{x\in X} \mathcal{A}(a,x) \end{array} \qquad\qquad X = F^{-1}\{Fa\}$$

$$\begin{array}{ccc} \mathcal{B}(Fa,b) \otimes \mathcal{B}(b,b') & \xrightarrow{\qquad \mu_{Fa,b,b'} \qquad} & \mathcal{B}(Fa,b') \\ {\scriptstyle \varphi_{a,b}\otimes \mathrm{id}}\downarrow & & \\ \Big(\displaystyle\sum_{x\in X} \mathcal{A}(a,x)\Big) \otimes \mathcal{B}(b,b') & & \\ {\scriptstyle \cong}\downarrow & & \\ \displaystyle\sum_{x\in X} \mathcal{A}(a,x) \otimes \mathcal{B}(Fx,b') & & \Big\downarrow{\scriptstyle \varphi_{a,b'}} \\ {\scriptstyle \sum \mathrm{id} \otimes \varphi_{x,b'}}\downarrow & & \\ \displaystyle\sum_{x\in X} \mathcal{A}(a,x) \otimes \Big(\sum_{y\in Y} \mathcal{A}(x,y)\Big) & & \\ {\scriptstyle \cong}\downarrow & & \\ \displaystyle\sum_{y\in Y}\sum_{x\in X} \mathcal{A}(a,x) \otimes \mathcal{A}(x,y) & \xrightarrow[\sum\sum \mu_{a,x,y}]{} & \displaystyle\sum_{y\in Y} \mathcal{A}(a,y) \end{array} \qquad\qquad \begin{array}{c} X = F^{-1}\{b\} \\ \\ Y = F^{-1}\{b'\} \end{array}$$

There are two surprises in the above definition: the distributivity requirement on the base of enrichment $\mathcal{V}$ and the indexing of the sum in the codomain of $\varphi_{a,b}$ by the fibre $F^{-1}\{b\}$ rather than all of $\operatorname{obj}(\mathcal{A})$. Both of these are essential to formulate the axiom expressing the compatibility of the cofunctor (F, φ) with the composition μ in $\mathcal{A}$ and $\mathcal{B}$.

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Proposition 2. *Let $(\mathcal{V}, \otimes, I)$ be an extensive category with a distributive monoidal structure. Then an enriched cofunctor $(F, \varphi)\colon \mathcal{A} \nrightarrow \mathcal{B}$ induces a span of enriched functors.*

Proof (sketch). Given an enriched cofunctor $(F, \varphi)\colon \mathcal{A} \nrightarrow \mathcal{B}$, define an enriched category $\mathcal{C}$ with the same objects as $\mathcal{A}$, and whose hom-objects are constructed by pullback along coproduct injections:

$$\begin{array}{ccc} \mathcal{C}(a, a') & \xrightarrow{Q_{a,a'}} & \mathcal{B}(Fa, Fa') \\ {\scriptstyle P_{a,a'}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \varphi_{a,Fa'}} \\ \mathcal{A}(a, a') & \xrightarrow[\iota_{a'}]{} & \sum_{x \in X} \mathcal{A}(a, x) \end{array} \qquad X = F^{-1}\{Fa'\}$$

The pullback projections are components of enriched functors $P\colon \mathcal{C} \to \mathcal{A}$ and $Q\colon \mathcal{C} \to \mathcal{B}$. □

Remark. Note that the enriched functor $P\colon \mathcal{C} \to \mathcal{A}$, defined in the proof above, is bijective on objects, while the enriched functor $Q\colon \mathcal{C} \to \mathcal{B}$ has the property that the morphism

$$[Q_{c,y}]\colon \sum_{y \in Y} \mathcal{C}(c, y) \longrightarrow \mathcal{B}(Qc, b) \qquad Y = Q^{-1}\{b\}$$

is invertible for each $c \in \mathrm{obj}(\mathcal{C})$ and $b \in \mathrm{obj}(\mathcal{B})$. These properties characterise amongst all spans of enriched functors those that correspond to enriched cofunctors.

Composition of enriched cofunctors is unital and associative, thus for each distributive monoidal category $(\mathcal{V}, \otimes, I)$ we obtain a category $\mathcal{V}$-$\mathcal{C}\mathrm{of}$ of enriched categories and enriched cofunctors. There is also a flat double category $\mathbb{C}\mathrm{of}(\mathcal{V})$ whose objects are enriched categories, whose horizontal morphisms are enriched functors, and whose vertical morphisms are enriched cofunctors.

Remark. If $\mathcal{V}$ has pullbacks and is extensive, then the category $\mathcal{V}$-$\mathcal{C}\mathrm{at}$ of enriched categories and enriched functors has pullbacks, and the double category $\mathbb{C}\mathrm{of}(\mathcal{V})$ has tabulators. Using these properties, the statement of Proposition 2 extends to the existence of a horizontally faithful double functor $\mathbb{C}\mathrm{of}(\mathcal{V}) \to \mathbb{S}\mathrm{pan}(\mathcal{V}\text{-}\mathcal{C}\mathrm{at})$ which implies that $\mathbb{C}\mathrm{of}(\mathcal{V})$ is *span representable* [7].

Definition 3. An *enriched lens* $(F, \varphi)\colon \mathcal{A} \rightleftharpoons \mathcal{B}$ consists of an enriched functor $F\colon \mathcal{A} \to \mathcal{B}$ and an enriched cofunctor $(F, \varphi)\colon \mathcal{A} \nrightarrow \mathcal{B}$ with the same underlying function $F\colon \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$ on objects, such that the following diagram commutes:

$$\begin{array}{ccccc} & & \sum_{x \in X} \mathcal{A}(a, x) & & \\ & {\scriptstyle \varphi_{a,b}}\nearrow & & \searrow{\scriptstyle [F_{a,x}]} & \\ \mathcal{B}(Fa, b) & & \xrightarrow[\mathrm{id}]{} & & \mathcal{B}(Fa, b) \end{array} \qquad X = F^{-1}\{b\}$$

Example 4. Let $\mathcal{V} = (\mathcal{S}\mathrm{et}, \times, 1)$. Then enriched categories and enriched cofunctors are ordinary categories and cofunctors, respectively. An enriched lens is precisely a *delta lens* [4].

Example 5. The category $\mathrm{w}\mathcal{S}\mathrm{et}$ of *weighted sets* [6] has objects given by sets A equipped with a weight function $|-|\colon A \to [0, \infty]$, and morphisms given by functions $f\colon A \to B$ such that $|fa| \leqslant |a|$ for all $a \in A$. It is a distributive monoidal category with $A \otimes B$ given by the set $A \times B$ with $|(a, b)| = |a| + |b|$, monoidal unit I given by the set $\{*\}$ with $|*| = 0$, and coproducts given by disjoint union. Categories and functors enriched in $\mathrm{w}\mathcal{S}\mathrm{et}$ are weighted categories and weighted functors. An enriched lens is precisely a *weighted lens*, recently introduced by Perrone [8].

Example 6. Consider the poset $([0, \infty], \geqslant)$ of extended non-negative real numbers as a category, with monoidal structure given by addition (such that $x + \infty = \infty$), and coproducts given by infima. A *Lawvere metric space* is an enriched category and an a *submetry* is an enriched lens.

Other bases of enrichment whose enriched categories, cofunctors, and lenses have interesting applied interpretations include:

- $(\mathcal{P}M, \subseteq, \cdot, \{e\})$, where $(M, \cdot, e)$ is a monoid and $\mathcal{P}$ is the powerset monad, whose enriched categories are closely related to generalised automata [2];
- $(\{-\infty\} \cup [0, \infty], \leqslant, +, 0)$ whose enriched categories are related to project scheduling [9].

Developing an enriched analogue of split opfibrations to describe *least-change* enriched lenses, and generalising the Grothendieck construction for enriched lenses are both topics of active research.

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